The Dialectica Translation of Type Theory

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TYPES
24th May 2016

Previously at TYPES...

Analytical description of the TYPES 2013 social event (Toulouse)

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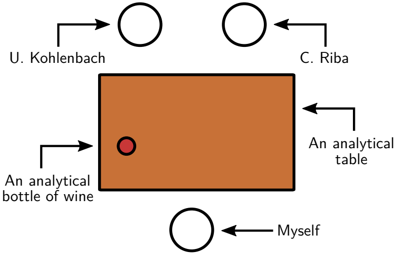

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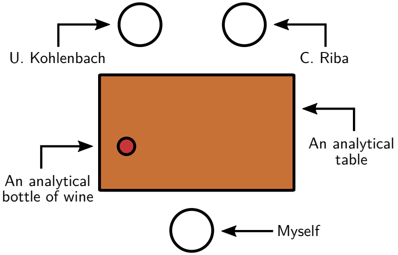

DRAMATIS PERSONAE:

- ULRICH KOHLENBACH, King of Dialectica
- COLIN RIBA, a Proof-Theory Gentleman
- PIERRE-MARIE PÉDROT, a Novice PhD Student
- THE BOTTLE OF WINE

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But I had found the matter for my PhD!

(Morale: you definitely should attend social events.)

A Quick Recap

- Dialectica is a logical translation due to Gödel
- Nowadays would be called a *realizability* intepretation

$$\vdash_{\mathbf{HA}} \pi : A \quad \rightsquigarrow \quad \begin{cases} \text{A } \lambda\text{-term } \pi^{\bullet} : [\![A]\!] \\ \text{A logical property } \pi^{\bullet} \Vdash A \text{ in the meta} \end{cases}$$

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- It preserves consistency, i.e. there is no $\pi : [\![\bot]\!]$ s.t. $\pi \Vdash \bot$
- It interprets strictly more than $\mathbf{HA}$, namely:

$$\begin{array}{ll} \mathrm{MP} : \neg(\forall x : \mathbb{N}. \neg P) \to \exists x : \mathbb{N}. P & (P \text{ decidable}) \\ \mathrm{IP} : (I \to \exists x : \mathbb{N}.P) \to \exists x : \mathbb{N}. I \to P & (I \text{ irrelevant}) \end{array}$$

Curry-Howard & Realizability

"Realizability interpretations tend to hide a programming translation."

Logic	Programming
Kreisel modified realizability	Identity translation
Krivine classical realizability	Lafont-Reus-Streicher CPS
Gödel Dialectica realizability	**?**

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Gödel Dialectica realizability	**A fancy one!**

- Gives first-class status to stacks
- Features a computationally relevant substitution
- Mix of LRS with delimited continuations
- Requires computational (finite) multisets $\mathfrak{M}$

Program translation, did you say?

- It operates on raw syntax (no need for the typing derivation)

$$t \in \Lambda \qquad \rightsquigarrow \qquad t^\bullet \in \Lambda + \ldots$$

- It preserves typing:

$$t : A \qquad \rightsquigarrow \qquad t^\bullet : [\![A]\!]$$

- **It preserves syntactic program equality (conversion)**:

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There is an itch: this requires multisets that compute definitionally

$$\emptyset \uplus t \equiv_\beta t \quad t \uplus u \equiv_\beta u \uplus t \quad (t \uplus u) \uplus r \equiv_\beta t \uplus (u \uplus r) \quad \ldots$$

Effectively

In CBN, the effect provided by Dialectica can be explained as follows:

$$\begin{array}{llcl} \text{From} & \lambda x.\, t & : & A \to B \\ & u & : & A \\ & \pi & : & B^{\perp} \\ \hline \text{Recover} & \mu & : & \mathfrak{M}\ A^{\perp} \end{array}$$

where:

- $X^{\perp}$ is the type of stacks accepting X (first-class contexts)
- μ is obtained by the following process:
  1. evaluate t on stack π
  2. each time t dereferences x, store the current stack ρ_i and continue with u
  3. when finished, return the multiset of all $[\rho_1; \ldots; \rho_n]$

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In practice

The translation goes roughly as follows:

$$A \text{ type} \quad \rightsquigarrow \quad \begin{cases} \mathbb{W}(A) \text{ witness type: type of objects} \\ \mathbb{C}(A) \text{ counter type: type of stacks} \end{cases}$$

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In particular,

$$\begin{array}{lcl} \mathbb{W}(A \to B) & := & (\mathbb{W}(A) \to \mathbb{W}(B)) \times (\mathbb{W}(A) \to \mathbb{C}(B) \to \mathfrak{M}\ \mathbb{C}(A)) \\ \mathbb{C}(A \to B) & := & \mathbb{W}(A) \times \mathbb{C}(B) \end{array}$$

There is a special translation handling open terms:

$$x_1 : \Gamma_1, \ldots, x_n : \Gamma_n \vdash t : A \quad \rightsquigarrow \quad \begin{cases} \mathbb{W}(\Gamma) \vdash t^\bullet : \mathbb{W}(A) \\ \mathbb{W}(\Gamma) \vdash t_{x_1} : \mathbb{C}(A) \to \mathfrak{M}\ \mathbb{C}(\Gamma_1) \\ \ldots \\ \mathbb{W}(\Gamma) \vdash t_{x_n} : \mathbb{C}(A) \to \mathfrak{M}\ \mathbb{C}(\Gamma_n) \end{cases}$$

Moar!

This translation is actually easily adapted to the dependent case.

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Interpreting dependent elimination through Dialectica looks complicated.

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`</End of the recap of my PhD>`

Decomposing Dialectica

- In her PhD, De Paiva gave a $\mathbf{LL}$ decomposition of Dialectica
- Root of double-glueing constructions
- Although it works, it inherits from the quirks of $\mathbf{LL}$

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- Although it works, it inherits from the quirks of $\mathbf{LL}$

- We give a new decomposition in $\mathbf{CBPV}$
- It is inherently highly dependent
- And it naturally provides an interpretation for the whole $\mathbf{CIC}$

CBPV

CBPV is a syntax for a pervasive class of models

$$
\begin{array}{llll}
\text{value types} & A, B & := & \mathbf{U}\,X \mid A + B \mid A \times B \mid \ldots \\
\text{computation types} & X, Y & := & \mathbf{F}\,A \mid A \to X \mid \ldots \\
\text{values} & v, w & := & \ldots \\
\text{computations} & t, u & := & \ldots
\end{array}
$$

Essentially, it decomposes Moggi's monadic language in an adjunction

$$T\,A := \mathbf{U}\,(\mathbf{F}\,A)$$

Thus, finer-grained.

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(We actually studied a dependently-typed variant, although not really thought about.)

Key idea of the decomposition

- We translate value and computation types alike
- The $\mathbb{C}(\cdot)$ type now crucially depends on a corresponding $\mathbb{W}(\cdot)$, i.e.

$$\mathbb{W}(A) : \square \qquad\qquad \mathbb{C}(A)[\cdot] : \mathbb{W}(A) \to \square$$

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$$\begin{array}{rcl}
\mathbb{W}(A \to X) & := & \Pi x : \mathbb{W}(A).\, \Sigma y : \mathbb{W}(X).\, (\mathbb{C}(X)[y] \to \mathfrak{M}\ \mathbb{C}(A)[x]) \\
\mathbb{C}(A \to X)[f] & := & \Sigma x : \mathbb{W}(A).\, \mathbb{C}(X)[\mathtt{snd}\ (f\ x)]
\end{array}$$

$$\begin{array}{rcl}
\mathbb{W}(\mathbf{F}\ A) & := & \mathbb{W}(A) \\
\mathbb{C}(\mathbf{F}\ A)[x] & := & \mathfrak{M}\ \mathbb{C}(A)[x]
\end{array}$$

$$\begin{array}{rcl}
\mathbb{W}(\mathbf{U}\ X) & := & \mathbb{W}(X) \\
\mathbb{C}(\mathbf{U}\ X)[x] & := & \mathbb{C}(X)[x]
\end{array}$$

Sequents

This naturally gives rise to the interpretation:

$$x_1 : \Gamma_1, \ldots, x_n : \Gamma_n \vdash t : X \quad \rightsquigarrow \quad \begin{cases} \mathbb{W}(\Gamma) \vdash t^\bullet : \mathbb{W}(X) \\ \mathbb{W}(\Gamma) \vdash t_{x_1} : \mathbb{C}(X)[t^\bullet] \to \mathfrak{M}\ \mathbb{C}(\Gamma_1)[x_1] \\ \ldots \\ \mathbb{W}(\Gamma) \vdash t_{x_n} : \mathbb{C}(X)[t^\bullet] \to \mathfrak{M}\ \mathbb{C}(\Gamma_n)[x_n] \end{cases}$$

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We never use the counter argument and merely pass it around!

In absence of datatypes, this is the **same** as the previous translation

Datatypes, at least

This counter dependency is only required for dependent elimination!

$$
\begin{array}{rcl}
\mathbb{W}(A \times B) & := & \mathbb{W}(A) \times \mathbb{W}(B) \\
\mathbb{C}(A \times B)[(x, y)] & := & \mathfrak{M}\ \mathbb{C}(A)[x] \times \mathfrak{M}\ \mathbb{C}(B)[y]
\end{array}
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\begin{array}{rcl}
\mathbb{W}(A + B) & := & \mathbb{W}(A) + \mathbb{W}(B) \\
\mathbb{C}(A + B)[\texttt{inl}\ x] & := & \mathfrak{M}\ \mathbb{C}(A)[x] \\
\mathbb{C}(A + B)[\texttt{inr}\ y] & := & \mathfrak{M}\ \mathbb{C}(B)[y]
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The argument is crucially observed through dependent elimination.
There is an implicit pattern-matching at the head of the definition.

An interesting remark

In absence of dependency, those types are emulated by being less precise. Typically, compare the dependent:

$$\begin{array}{rcl} \mathbb{C}(A \times B)[(x, y)] & := & \mathfrak{M}\ \mathbb{C}(A)[x] \times \mathfrak{M}\ \mathbb{C}(B)[y] \\ \mathbb{C}(A + B)[\texttt{inl}\ x] & := & \mathfrak{M}\ \mathbb{C}(A)[x] \\ \mathbb{C}(A + B)[\texttt{inr}\ y] & := & \mathfrak{M}\ \mathbb{C}(B)[y] \end{array}$$

with the $\mathbf{LL}$-induced:

$$\begin{array}{rcl} \mathbb{C}(A \times B) & := & \mathbb{W}(A) \times \mathbb{W}(B) \to \mathfrak{M}\ \mathbb{C}(A) \times \mathfrak{M}\ \mathbb{C}(B) \\ \mathbb{C}(A + B) & := & (\mathbb{W}(A) \to \mathfrak{M}\ \mathbb{C}(A)) \times (\mathbb{W}(B) \to \mathfrak{M}\ \mathbb{C}(B)) \end{array}$$

Exploiting the translation

There exists a Dialectica translation from $\mathbf{CIC}$ into $\mathbf{CIC} + \mathfrak{M}$.

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- Looks like their theory is decidable (?)
- Maybe we can implement a type-checker (?)

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- Maybe we can implement a type-checker (?)

In any case, we can't reuse an off-the-shelf implementation of type theory.

A funny, more intensional $\mathbf{CIC}$ (in CBN)

Through the translation, we get strictly more than $\mathbf{CIC}$.

$\mathbf{CIC}^D$ negates functional extensionality (and thus univalence):

$$\mathbf{CIC}^D \vdash \neg(\Pi f\ g.\,(\Pi x.\, f\ x = g\ x) \to f = g)$$

It is fairly trivial, because of the second component of arrows. E.g.:

$$\lambda\_.\,() \cong 0 \quad \text{vs.} \quad \lambda().\,() \cong 1 \quad \text{in} \quad 1 \to 1 \cong \mathbb{N}$$

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Yet it is not that badly behaved w.r.t. functions:

$\mathbf{CIC}^D$ preserves η-expansion.

Towards implicit complexity in Type Theory?

A generalization of the previous constatation:

$\mathbf{CIC}^{D}$ allows to count the uses of a function argument.

Indeed, the size of the multisets corresponds to the number of uses.

- It is not trivial, because very higher-order-ish
- In particular, the number of uses depends on the argument

E.g.: $\lambda b : \mathbb{B}.\, \mathtt{if}\ b\ \mathtt{then}\ ()\ \mathtt{else}\ \mathtt{if}\ b\ \mathtt{then}\ ()\ \mathtt{else}\ ()$

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Can we use it to implement implicit complexity in $\mathbf{CIC}^D$?

Food for thought

- Why is Dialectica inherently dependent?
- Can $\mathbf{LL}$ be encoded with dependent elimination?
- Does the implicit complexity stuff really requires multisets?
- How much can we fiddle with the $\mathbf{CBPV}$ decomposition?
- Can we merge $\mathbf{CBPV}$ and $\mathbf{LL}$?

Thanks for your attention.