From Lost to the River: Embracing Sort Proliferation

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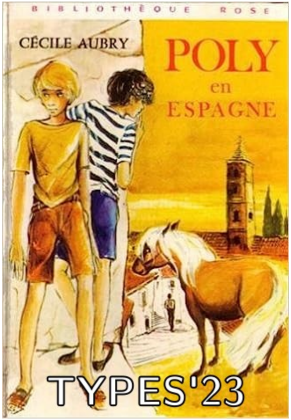

Types, types, types

Type theory is about types!

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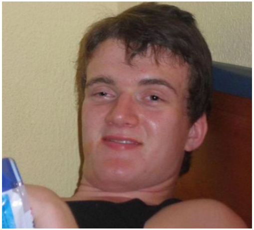

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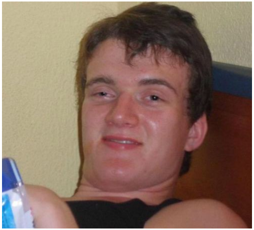

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In MLTT and its variants, types are **also** terms.

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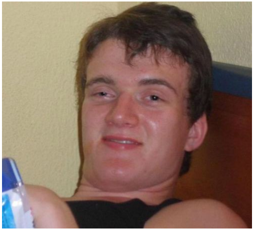

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The type of a type is a universe.

$$\vdash A : \mathtt{Type} \qquad \sim \qquad A \text{ is a type}$$

Quis custodiet ipsos custodes?

What is the type of the universe?

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Martin-Löf '71

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MLTT with the above rule is inconsistent.

Standard solution

$$\vdash \mathtt{Type}_n : \mathtt{Type}_{n+1} \qquad \text{for } n \in \mathbb{N}$$

Meanwhile in the Real World™

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- You have to explicitly set a level n for every single `Type`
- If you need one intermediate universe in the middle of a proof...

  $\rightsquigarrow$ Better have used BASIC line numbering!
- Martin-Löf forbid that you want to use a term at two different levels

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It's a very **anti-modular** feature!

Well-know issues imply well-known solutions

Thankfully, it's a well-known issue

1. Floating universes aka (global) algebraic constraints

$$\mathfrak{S} \vDash i < j$$

2. Some flavour of universe polymorphism aka bound levels

$$\vdash M : \tilde{\forall} i j.\ \mathtt{Type}_i \to \mathtt{Type}_{j+1}$$

3. More exotic stuff: TEMPLATE POLY, crude but effective...

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Not mutually exclusive! Coq uses 1 + 2 + TEMPLATE POLY.

Unexpected Prequel

Modularity is restored, the Galaxy is at peace...

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BREAKING NEWS

Nope! There are alternate universes out there.

Rising from the CIC tradition, we had `Prop` for decades in Coq.

Propping up the Scene

Prop: a mishmash of features

- Impredicative: $\Pi x : A.\, B : \texttt{Prop}$ as long as $B : \texttt{Prop}$
- 100% compatible with proof-irrelevance (but not irrelevant)
- Erasable through extraction

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- Prop to Prop is fine
- Prop to Type must satisfy **singleton elimination**

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Due to these features, elimination of `Prop` inductives is tricky

- `Prop` to `Prop` is fine
- `Prop` to `Type` must satisfy **singleton elimination**

This prevents (naive) universe polymorphism over `Prop`

$$\texttt{Inductive Box (A : Type}_i\texttt{) : Type}_j \texttt{ := box : A} \rightarrow \texttt{Box A}$$

It's only the beginning

We have to duplicate everything between `Prop` and `Type`

- All stdlib basic inductives come in two flavours (e.g. $\exists$ vs. Σ)
- Mitigated by `Prop` $\subseteq$ `Type` $\quad\rightsquigarrow\quad$ only the return sort matters
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A new opponent has appeared

... but things went **really south** since the introduction of `SProp`

- Now we need at least three variants
- ... but `SProp` $\not\subseteq$ `Type` $\quad\rightsquigarrow\quad$ 2^n variants for n parameters
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So long for modularity!

What now?

There is no reason to stop at three hierarchies.

- Fibrant vs. strict universes (HoTT)
- Pure vs. impure universes (Exceptional Theory, MTT, ...)
- Setoid / parametric / cubic / blue-haired ω-potatoid universes

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Resistance is futile. Let's embrace sort proliferation!

with the dark powers of sort polymorphism

All universes in a single proof assistant

Land of the Free

It's a Revolution

Variables!

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$$
\begin{array}{lcll}
q & ::= & \alpha \mid \mathbf{Type} \mid \mathbf{Prop} \mid \mathbf{SProp} \mid \ldots & \text{(sort qualities)} \\
M & ::= & \mathtt{Sort}_{\ell}^{q} \quad (\ell \text{ level}, q \text{ quality}) \mid \ldots & \text{(terms)}
\end{array}
$$

- No algebraic structure on qualities
- Usual universes become notations

$$
\mathtt{Type}_{\ell} := \mathtt{Sort}_{\ell}^{\mathbf{Type}} \qquad \mathtt{Prop} := \mathtt{Sort}_{0}^{\mathbf{Prop}} \qquad \mathtt{SProp} := \mathtt{Sort}_{0}^{\mathbf{SProp}}
$$

Give me the Rules

$$\frac{}{\vdash \mathtt{Sort}^{q}_{\ell} : \mathtt{Type}_{\ell+1}}$$

$$\frac{}{\vdash \mathtt{Sort}^{\mathbf{Prop}}_{i} \equiv \mathtt{Sort}^{\mathbf{Prop}}_{j}} \qquad \frac{}{\vdash \mathtt{Sort}^{\mathbf{SProp}}_{i} \equiv \mathtt{Sort}^{\mathbf{SProp}}_{j}}$$

$$\frac{\vdash A : \mathtt{Sort}^{q_A}_{\ell_A} \qquad x : A \vdash B : \mathtt{Sort}^{q_B}_{\ell_B}}{\vdash \Pi x : A.\, B : \mathtt{Sort}^{q_B}_{\ell_A \vee \ell_B}}$$

- `Type` always classifies sorts
- Impredicativity implemented as level-irrelevance
- Product rule is call-by-name-ish / impredicative-friendly

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These rules are compatible with all our intended instances

Qot qot quantum

We extend Coq universe polymorphism with sort polymorphism.

$$\mathbf{c} : \forall(\alpha_1 \ldots \alpha_n \in q).\, \forall(i_1 \ldots i_m \in \ell \mid \mathfrak{S}).\, A$$

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- Prenex, external polymorphism
- Generalization of universe polymorphism, compatible with it
- No constraint system on sorts! (†)

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We have a scheme that is compatible with non-sort polymorphic code.

- Introduction rules for inductive types are unchanged.
- The tricky part is elimination: we generalize singleton criteria

$M : \mathcal{I} : \mathtt{Sort}^{q}$ and $T : \mathtt{Sort}^{r} \vdash \mathtt{case}\ M\ \mathtt{return}\ T\ \mathtt{with}\ \ldots$ allowed?

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q	r
α	$\{\alpha\}$
Type	any
Prop	any if $\mathcal{I}$ singleton (finite quality check) $\{\mathtt{Prop}, \mathtt{SProp}\}$ otherwise
SProp	any if $\mathcal{I}$ empty $\{\mathtt{SProp}\}$ otherwise

Final product

TYPE THEORY IS AT PEACE.

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- Sort-poly is a conservative extension (like univ-poly)
- ... just a glorified copy-paste!
- In particular, it doesn't change the consistency of the ambient theory
- It blends easily with univ-poly

Proof.

All typing rules are stable by substitution with ground qualities. □

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TEMPLATE POLY

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We can now model it with cumulative inductives + sort-poly!

Template-poly inductives are:

- polymorphic cumulative types whose universe levels are all irrelevant
- levels must satisfy a syntactic “inferability” criterion
- template parameters are sort-poly with the same sort as inductive

TL;DR: TEMPLATE POLY has an intended semantics (†)

What is to be done

The sort-poly infrastructure is now part of Coq **unification** (8.17).

- Part of the kernel term representation
- Used to delay sort assignment in universe unification
- Dedicated handling of $\texttt{Prop} \subseteq \texttt{Sort}^q \subseteq \texttt{Type}$
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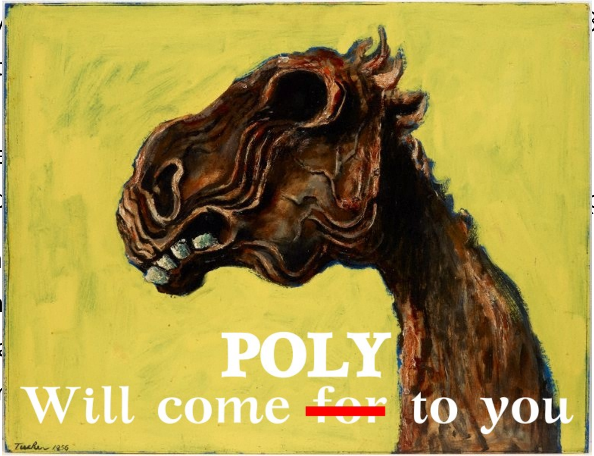
The sort-poly  .17).

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Scribitur ad narrandum, non ad probandum

Thanks for your attention.