Effects, Substitution and Induction

An Explosive Ménage à Trois

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INRIA

TYPES 2019

13th June 2019

It's Time to CIC Ass

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You fire your favourite IDE

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COULD YOU WRITE A HELLO WORLD?

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HOW DO I REASON CLASSICALLY?

Thesis

We want a type theory with effects!

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To program more!

Non-termination

Exceptions

State...

To prove more!

Classical logic

Univalence

Choice...

Something is Rotten in the State of Type Theory

Classical logic does not play well with type theory.

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Effectful theories are always half-broken

dependent elimination has to be restricted (BTT)
or consistency forsaken, or worse

The Important Bit

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Herbelin / Barthe-Uustalu results are instances of a generic phenomenon!

Also, this is kind of folklore.

... but I don't recall reading it formally anywhere.

Effects, Effects Everywhere!

Definition

A type theory has *observable effects* if there is a closed term $t : \mathbb{B}$ that is **not observationally equivalent to a value**, i.e. there is a context $C[\cdot]$ s.t.

$$C[\texttt{true}] \equiv \texttt{true} \quad \text{and} \quad C[\texttt{false}] \equiv \texttt{true} \quad \text{but} \quad C[t] \equiv \texttt{false}$$

This happens for many kind of effects, including continuations.

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Such terms are typically called **non-standard** booleans.

e.g. a function $\texttt{is\_empty} : \Pi A.\, A \rightarrow \mathbb{B}$ deciding inhabitation of a type.

A Tension Build-up

Definition

A type theory enjoys *substitution* if the following rule is derivable.

$$\frac{\Gamma, x : X \vdash \bullet : A \qquad \Gamma \vdash t : X}{\Gamma \vdash \bullet : A\{x := t\}}$$

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Substitution is usually taken for granted

... hint: this is a bias

Almost There

Definition

A type theory enjoys *dependent elimination* on booleans if we have:

$$\frac{\Gamma, b : \mathbb{B} \vdash P : \square \qquad \Gamma \vdash \bullet : P\{b := \texttt{true}\} \qquad \Gamma \vdash \bullet : P\{b := \texttt{false}\}}{\Gamma, b : \mathbb{B} \vdash \bullet : P}$$

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The landmark of dependent type theory, used to encode induction!

Absence of dependent elimination smells of trivial theories.

« Ciel, mon mari ! »

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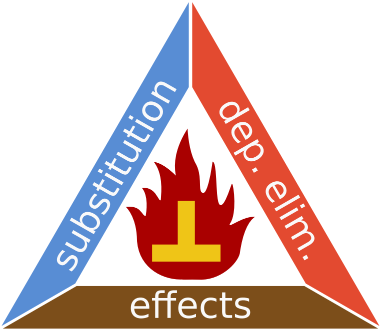

Theorem (Explosive Ménage à Trois a.k.a. Fire Triangle)

Effects + substitution + dep. elimination $\vdash$ *logically inconsistent.*

There Is No Free Lunch

The proof is actually straightforward.

Proof.

If C distinguishes boolean values from an effectful term M, prove by dependent elimination $\Pi(b : \mathbb{B}).\ C[b] = \texttt{false}$, apply to M and derive $\texttt{true} = \texttt{false}$. $\square$

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We essentially retrofitted the definition of effects to make it work.

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So it’s not cheating either.

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And now for a high-level overview of the problem and solutions

It is not a Bug, it is a Feature™

Dependency entails one major difference with usual type systems.

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Meet conversion:

$$\frac{A \equiv_\beta B \qquad \Gamma \vdash M : B}{\Gamma \vdash M : A}$$

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Bad news 1

Typing rules embed the dynamics of programs!

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Bad news 1

Typing rules embed the dynamics of programs!

Combine that with this other observation and we're in trouble.

Bad news 2

Effects make reduction strategies relevant.

Call-by-name vs. Call-by-value

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Call-by-name: **functions** well-behaved vs. **inductives** ill-behaved

Call-by-value: **inductives** well-behaved vs. **functions** ill-behaved

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In **call-by-name** + effects:

$$(\lambda x.\, M)\; N \equiv M\{x := N\} \quad \rightsquigarrow \quad \textbf{arbitrary substitution}$$
$$(\lambda b : \texttt{bool}.\, M)\; \mathbf{fail} \quad \rightsquigarrow \quad \textbf{non-standard booleans}$$

Substitution is a feature of call-by-name

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Substitution is a feature of call-by-name

In **call-by-value** + effects:

$$(\lambda x.\, M)\ V \equiv M\{x := V\} \quad \rightsquigarrow \quad \textbf{substitute only values}$$
$$(\lambda b : \mathbb{B}.\, M)\ N \equiv (\lambda b : \mathbb{B}.\, M)\ V \quad \rightsquigarrow \quad \textbf{boolean values are booleans}$$

Dependent elimination is a feature of call-by-value

Impossible is not French

Three knobs $\Rightarrow$ **Four** solutions

Impossible is not French

Three knobs ⇒ **Four** solutions

- **Down with effects**: CBN and CBV reconcile

This is good ol' CIC, Keep Calm and Carry on.

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- **Go CBN** and restrict dependent elimination: Baclofen Type Theory

$$\texttt{if}\ M\ \texttt{then}\ N_1\ \texttt{else}\ N_2 : \texttt{if}\ M\ \texttt{then}\ P_1\ \texttt{else}\ P_2$$

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- **CBV rules**, respect values, and dump substitution: one weird trick

The least conservative approach

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The least conservative approach

- Who cares about consistency? **I want all!**

A paradigm shift: from type theory to dependent languages, e.g. ExTT

A Generic Workaround

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$$\partial\mathsf{CBPV}$$

(We had to pick a fancy name.)

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∂CBPV

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The main novelties: two for the price of one

- Not one, but **two** parallel hierarchies of universes: $\square_v$ vs. $\square_c$!
- Not one, but **two** let-bindings!

$$\frac{\Gamma \vdash t : F\ A \qquad \Gamma \vdash X : \square_c \qquad \Gamma, x : A \vdash u : X}{\Gamma \vdash \texttt{let}\ x := t\ \texttt{in}\ u : X}$$

$$\frac{\Gamma \vdash t : F\ A \qquad \Gamma, x : A \vdash X : \square_c \qquad \Gamma, x : A \vdash u : X}{\Gamma \vdash \texttt{dlet}\ x := t\ \texttt{in}\ u : \texttt{let}\ x := t\ \texttt{in}\ X}$$

- Justified by all of our syntactic models so far (and we have quite a few)

Many More

This was a very high-level talk

Many things I did not discuss here!

- A good notion of purity: thunkability vs. linearity
- Complex ∂CBPV encodings
- Presheaves as observationally pure terms of an impure CBV theory

`http://pédrot.fr/articles/dcbpv.pdf`

Conclusion

- Effects and dependent types: choose your side.
  $\rightsquigarrow$ Purity, CBN, CBV, Michael Bay?
- Even inconsistent theories have practical interest.

Thanks for your attention.