Dialectica the Ultimate

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INRIA

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In Media Res

Today's Menu

The most tricky question that I have been tormented with for a while

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TLLA :≡ Trends in **Linear Logic** and Applications

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$$\text{TLLA} :\equiv \text{Trends in } \textbf{Linear Logic} \text{ and Applications}$$

(It is still time to escape.)

How do you do, fellow linear logicians?

Linear Logic?

- LL was fashionable as an undergraduate
- We had a group discussing the latest GoI trends and stuff
- I betrayed linear logic right during my PhD (defended in 2015)
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« Seriously, what I am going to talk about at TLLA? »

How I Learned to Stop Worrying and Love the Bang

The universal solution to the TLLA problem

(Hint: it's actually in the title.)

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- Shameless recycling: this was the topic of my PhD
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- Latent schizophrenia: it keeps appearing everywhere (and nobody sees it)

A Short, Mostly Wrong History of Dialectica

Dialectica in a nutshell

- The great ancestor of realizability
- The name comes from the journal it was published in
- Basically a fancy realizability model of HA^ω into System T
- Designed by Gödel in the 30's but published in 1958
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In *Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes*, Gödel passes more time discussing α-conversion than the actual Dialectica interpretation.

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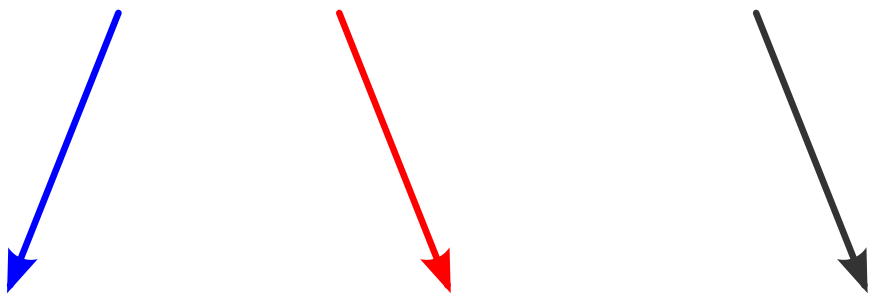
(You have been warned.)

Anatomy of a Dusty Realizability

If $\mathsf{HA}^\omega \vdash A$ then

$$\exists \vec{u}.\ \forall \vec{x}.\ A_D[\vec{u}, \vec{x}]$$



$\mathbb{W}(A)$	$\mathbb{C}(A)$	$\bot_A$
Sequence of types	Sequence of types	$\mathbb{W}(A) \to \mathbb{C}(A) \to \mathtt{Prop}$
Witness	Counter	Orthogonality

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Legitimate question

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$$\begin{aligned} \mathbb{W}(A \wedge B) &= \mathbb{W}(A) \times \mathbb{W}(B) \\ \mathbb{W}(A \vee B) &= \mathbb{W}(A) + \mathbb{W}(B) \end{aligned}$$

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$$A \to B$$

Implying this Makes Sense

There is more to arrows than just functions!

$$\mathbb{W}(A \to B) = \begin{cases} \mathbb{W}(A) \Rightarrow \mathbb{W}(B) \\ \mathbb{W}(A) \Rightarrow \mathbb{C}(B) \Rightarrow \mathbb{C}(A) \end{cases}$$

$$\mathbb{C}(A \to B) = \mathbb{W}(A) \times \mathbb{C}(B)$$

Functions have both a **forward** and a **backward** component.

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A clear reminiscence of LL!

$$!A \multimap B \quad \cong \quad B^{\perp} \multimap ?A^{\perp}$$

Beating a Dead Horse

Since De Paiva, it is well-known that Dialectica factorizes through LL

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Reminder:

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Can Dialectica Break Bricks?

Actually, most* models of LL are Dialectica-like

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Empirical Observation

Dialectica is the most natural way to linearize an intuitionistic calculus

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Empirical Observation

Dialectica is the most natural way to linearize an intuitionistic calculus

(∗) except maybe games, and even there if we squint enough there are weird similarities.

The Horse that Did Not Stop Beating Itself to Death

Dialectica keeps popping everywhere

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- Some of these Dialectica-like constructs are well-known
- Some are more obscure, e.g. sequential algorithms (cf. some workshop in Marseille)
- Surprisingly, **obvious** *but* **deep** connections were only made recently

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This afternoon at LICS: Dialectica actually computes differentials

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My now systematic reaction when faced with something related to LL

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Graded / quantitative types are a poor man's Dialectica

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Graded / quantitative types are a poor man's Dialectica

- More positively, Dialectica is a finer kind of graded types
- Compatible with rich types (i.e. MLTT)
- Dialectica as proof-relevant, higher-order complexity annotations

Part I
Dialectica Done Right

Curry-Howard to the Rescue

Let us present Dialectica as a program translation!

- Without cheating too much
- A rich language in the source: MLTT
- The target will also be a variant of MLTT (some cheating involved here)
- A variant of Moss-von Glehn model (a.k.a the Google Drive note by A. Bauer and myself)

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... but *why*?

- To get rid of most of Gödel's hacks
- To care about the equational theory
- Because dependent types are cool, expressive and effective

MLTT for Dummies

Do not panic

I am not expecting you to be experts in dependent types

$\rightsquigarrow$ MLTT is just a fancily-typed λ-calculus. (Any Agda or Coq user around?)

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Otherwise

The major differences with your run-of-the-mill λ-calculus.

- Types can depend on terms (duh!)

$$\Pi(x : A).\, B$$

- The equational theory on terms lifts to the type layer

$$\texttt{refl} : \Pi(A : \texttt{Type})(x : A).\, x = x \qquad \rightsquigarrow \qquad (\texttt{refl}\ \mathbb{N}\ 2) : 1 + 1 = 2$$

- Large elimination: build types by pattern-matching on positives

$$\lambda(b : \mathbb{B}).\, \texttt{if}\ b\ \texttt{then}\ \mathbb{N}\ \texttt{else}\ (\mathbb{N} \to \mathbb{N}) \quad : \quad \mathbb{B} \to \texttt{Type}$$

Simpler? It Depends

Remember that in standard Dialectica, a type A is converted to

a type $\mathbb{W}(A)$ a type $\mathbb{C}(A)$ a relation $\perp_A \subseteq \mathbb{W}(A) \times \mathbb{C}(A)$

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Thanks to the added expressivity, we can conflate $\mathbb{C}(A)$ and $\perp_A$

An MLTT type A will be converted to

- a type $\mathbb{W}(A)$
- a dependent type $\mathbb{C}(A) : \mathbb{W}(A) \to \mathtt{Type}$

No need for orthogonality, it is built into $\mathbb{C}$!

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Remark: for readability I will write $\mathbb{C}(A)\langle M \rangle$ for $\mathbb{C}(A)\ M$.

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We must validate the equational theory of MLTT

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We must introduce an additional structure $\mathfrak{M}$ called **abstract multisets**

- A generalization of finite multisets
- Clearly there to handle exponentials
- Bear with me...

Categorical Nonsense

An abstract multiset structure is just a semiringoid!

- A monad $\mathfrak{M}$ with return and bind:

$$\{\cdot\} : A \to \mathfrak{M}\ A \qquad \gg\!= : \mathfrak{M}\ A \to (A \to \mathfrak{M}\ B) \to \mathfrak{M}\ B$$

- that further has a commutative monoid structure:

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If you forget the parameter, equations are those of a semiring

$$\{M\} \gg\!= F \equiv F\ M \qquad M \gg\!= (\lambda x.\, \{x\}) \equiv M$$

$$(M \gg\!= F) \gg\!= G \equiv M \gg\!= (\lambda x.\, F\ x \gg\!= G)$$

$$\varnothing \oplus M \equiv M \qquad M \oplus \varnothing \equiv M \qquad (M \oplus N) \oplus P \equiv M \oplus (N \oplus P)$$

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Prototypical example: finite multisets

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For **each** $x : X \in \Gamma$, a **reverse** translation:

$$\mathbb{W}(\Gamma) \vdash [M]_x : \mathbb{C}(A)\langle [M] \rangle \to \mathfrak{M}\ \mathbb{C}(X)\langle x \rangle$$

Reminder:

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If $\|\Gamma\| = n$, this means I have $n + 1$ objects!

Motto

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In particular,

- $[x]_x := \lambda\pi.\,\{\pi\}$ (dereliction)
- $[y]_x := \lambda\pi.\,\varnothing$ if $x \neq y$ (weakening)

noting that

$$[x] := x$$

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Substitution lemma (contraction + promotion)

$$[M\{x := N\}]_y\ \pi \quad \equiv \quad ([M]_y\{x := [N]\}\ \pi) \ \oplus\ ([M]_x\{x := [N]\}\ \pi \gg\!= [N]_y)$$

A Functional Functional Interpretation

There is no structure in our theory yet, let's look at functions!

$$\mathbb{W}(\Pi(x : A).\, B) \qquad := \left\{ \begin{array}{l} f : \Pi(x : \mathbb{W}(A)).\, \mathbb{W}(B) \\ \varphi : \Pi(x : \mathbb{W}(A)).\, \mathbb{C}(B)\langle f\ x\rangle \to \mathfrak{M}\ \mathbb{C}(A)\langle x\rangle \end{array} \right\}$$

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Forward translations are easy.

$$\begin{array}{lcl} [\lambda x.\, M] & := & (\lambda x.\, [M]), (\lambda x.\, [M]_x) \\ [M\ N] & := & [M].1\ [N] \end{array}$$

Recall: if $\Gamma \vdash M : A$ then $\mathbb{W}(\Gamma) \vdash [M]_x : \mathbb{C}(A)\langle [M]\rangle \to \mathfrak{M}\ \mathbb{C}(X)\langle x\rangle$ provided $x : X \in \Gamma$

Full Reverse

Reverse translations are quite a mouthful.

$$[\lambda x.\, M]_x := \lambda(x, \pi).\, [M]_x\ \pi$$

$$[M\ N]_x := \lambda\pi.\quad ([M].2\ [N]\ \pi)\ \oplus\ ([M]_x\ ([N], \pi) \gg\!= [N]_y)$$

Cheat Sheet

1. If $\Gamma \vdash M : A$ then $\mathbb{W}(\Gamma) \vdash [M]_x : \mathbb{C}(A)\langle [M] \rangle \to \mathfrak{M}\ \mathbb{C}(X)\langle x \rangle$ provided $x : X \in \Gamma$.
2. $\mathbb{W}(\Pi(x : A).\, B) := \left\{ \begin{array}{l} f : \Pi(x : \mathbb{W}(A)).\, \mathbb{W}(B) \\ \varphi : \Pi(x : \mathbb{W}(A)).\, \mathbb{C}(B)\langle f\ x \rangle \to \mathfrak{M}\ \mathbb{C}(A)\langle x \rangle \end{array} \right\}$
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Cheat Sheet

1. If $\Gamma \vdash M : A$ then $\mathbb{W}(\Gamma) \vdash [M]_x : \mathbb{C}(A)\langle [M] \rangle \to \mathfrak{M}\ \mathbb{C}(X)\langle x \rangle$ provided $x : X \in \Gamma$.
2. $\mathbb{W}(\Pi(x : A).\, B) := \left\{ \begin{array}{l} f : \Pi(x : \mathbb{W}(A)).\, \mathbb{W}(B) \\ \varphi : \Pi(x : \mathbb{W}(A)).\, \mathbb{C}(B)\langle f\ x \rangle \to \mathfrak{M}\ \mathbb{C}(A)\langle x \rangle \end{array} \right\}$
3. $\mathbb{C}(\Pi(x : A).\, B)\langle f, \varphi \rangle := \Sigma(x : \mathbb{W}(A)).\, \mathbb{C}(B)\langle f\ x \rangle$

Application basically follows the substitution lemma.

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$\rightsquigarrow$ term translation follows the leading motto

- Variable use in $\texttt{inl}\ M$ and $\texttt{inr}\ M$ is obvious
- Variable use in

$$\texttt{case}\ M\ \texttt{with}\ \texttt{inl}\ x \to N_1 \mid \texttt{inr}\ y \to N_2$$

comes from either M or N_i depending on M

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A lot of somewhat arbitrary choices

Trust me

We have defined a model of MLTT.

If $\Gamma \vdash M : A$,

- $\mathbb{W}(\Gamma) \vdash [M] : \mathbb{W}(A)$
- $\mathbb{W}(\Gamma) \vdash [M]_x : \mathbb{C}(A)\langle [M] \rangle \to \mathfrak{M}\ \mathbb{C}(X)\langle x \rangle$ for all $x : X \in \Gamma$

and similarly for conversion.

Part II

A Resourceful Take

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- for efficiency
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As a first-order approximation I will lump them together here

- I will gloss over the differences
- and deliberately ignore some technicalities

Get in the Ring

Typically, fix a **resource** semi-ring $(\mathbb{M}, +, \times)$.

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Simplest example is $0 \leq 1 \leq \omega$.

Varying Variables

Variables are annotated by the semi-ring

$$x_1 :_{\alpha_1} X_1, \ldots x :_{\alpha_n} X_n \vdash M : A$$

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We should concentrate on the big picture!

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Types and terms are annotated accordingly

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Sprinkle with subtyping induced by $\leq$

Apply Copiously

More interesting is the case of application!

$$\frac{\Gamma \sim \Delta \qquad \Gamma \vdash M : \Pi(x :_\alpha A).\, B \qquad \Delta \vdash N : A}{\Gamma + (\alpha \times \Delta) \vdash M\ N : B\{x := N\}}$$

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This is really where the semi-ring structure shines!

The Sad Truth

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Let me give some examples of increasing annoyance

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How to type:

$$\texttt{if } M \texttt{ then } N_1 \texttt{ else } N_2$$

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Both are overapproximations: it actually depends on M

$$\texttt{if } M \texttt{ then } (\texttt{if } M \texttt{ then } N_1 \texttt{ else } N_2) \texttt{ else } N_3$$

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This was the easy one! There are only two branches for $\mathbb{B}$.

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Both solutions are quite bad.

I Want My Functions Back

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$$(A \to_\alpha B) \to_\beta C$$

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What happens in practice: (a.k.a. fancy subtyping doesn't work)

- $0 \rightsquigarrow$ runtime irrelevant terms (e.g. types)
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- $\omega \rightsquigarrow$ all other cases

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You have just reinvented a single runtime irrelevant modality

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... but in my opinion these are very contrived.

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DIALECTICA

(what a surprise!)

The Main Idea

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For this, we need to turn $x :_\alpha X$ into something that looks like

$$\Gamma \vdash M : A \quad \text{and} \quad x : X \in \Gamma \quad \rightsquigarrow \quad \alpha_x(M) : \mathcal{O}_A^\Gamma(X) \to \mathbb{M}$$

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Wait, this is a Déjà Vu

$$\Gamma \vdash M : A \text{ and } x : X \in \Gamma \ \rightsquigarrow \ \mathbb{W}(\Gamma) \vdash [M]_x : \mathbb{C}(A)\langle [M] \rangle \to \mathfrak{M}\ \mathbb{C}(X)\langle x \rangle$$

A Bug In the Semi-Ring Matrix

What has been seen cannot be unseen

$$[x]_y\ \pi := \varnothing \qquad [x]_x\ \pi := \{\pi\}$$

$$[\lambda y.\ M]_x\ (y, \pi) := [M]_x\ \pi$$

$$[M\ N]_x\ \pi := \begin{array}{c} [M]_x\ ([N], \pi) \\ \oplus \\ ([M].2\ [N]\ \pi \gg\!= [N]_x) \end{array}$$

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Dialectica, Dialectica Everywhere

Dialectica is the HO dependent graded type of a term

- The semiring is replaced by its oidification (a monad + comm. add. monoid)
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This solves the graded expressivity issue

$\rightsquigarrow$ When pattern-matching over $M : \mathbb{B}$, $\alpha \sqcup \beta$ becomes basically

$$\texttt{if } M \texttt{ then } \alpha \texttt{ else } \beta$$

$\rightsquigarrow$ The argument of a function f has a now a highly dynamic annotation

$$\varphi : \Pi(x : \mathbb{W}(A))(\pi : \mathbb{C}(B)\langle f\ x\rangle).\, \mathfrak{M}\ \mathbb{C}(A)\langle x\rangle \qquad \text{(before: a fixed } \mathbb{M}\text{)}$$

So What?

I don't know what to do of this

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What do you think about all this?

$\rightsquigarrow$ this is a call for help!

Thanks for your attention.