An Effectful Way to Eliminate Addiction to Dependence

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The Most Important Issue of Them All

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- Assume you want to show the wonders of Coq to a fellow programmer
- You fire your favourite IDE
- ... and you're asked the *dreadful* question.
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**COULD YOU WRITE A HELLO WORLD PROGRAM PLEASE?**
A Well-known Limitation

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- no exceptions
- no state
- no non-termination
- no printing
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**Intuitionistic Logic ↔ Functional Programming**

which means **no effects** in TT, amongst which:

- no exceptions
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- no non-termination
- no printing
- ... and thus no Hello World!
In less expressive settings, a few workarounds are known.

Typically, on the programming side, use the monadic style.

- A type $T : \square \rightarrow \square$
- A combinator $\text{return} : \alpha \rightarrow T\alpha$
- A combinator $\text{bind} : T\alpha \rightarrow (\alpha \rightarrow T\beta) \rightarrow T\beta$
- A few equations
On Burritos

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Interpret mechanically effectful programs using this (see Moggi).

This is pervasive in e.g. Haskell.
On the logic side, take the issue the other way around.
Less is More

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Effects are known to implement non-intuitionistic axioms!

- callcc \(\leadsto\) classical logic (Griffin '90)
- exceptions \(\leadsto\) Markov's rule (Friedman's trick)
- global monotonous cell \(\leadsto\) \(\neg\)CH (forcing)
- delimited continuations \(\leadsto\) double negation shift
- ...

Achieve this using logical translations, e.g. double-negation.
We want a type theory with effects!

1. To program more (exceptions, non-termination...)
2. To prove more (classical logic, univalence...)

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2. To prove more (classical logic, univalence...)
3. To write Hello World.
Problem is:

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  \[ \text{dbind} : \Pi \hat{x} : T \alpha. (\Pi x : \alpha. T (\beta x)) \to T (\beta ?) \]

- They don't acknowledge types-as-terms either

- And they don't preserve the computational rules of TT
The Expressivity Wall

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- Monads do not acknowledge dependence
  
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- They don't acknowledge types-as-terms either
- And they don't preserve the computational rules of TT

On the other hand:
- Herbelin showed that CIC + callcc is unsound!
In This Talk

1. Adding a vast range of effects to (almost) full TT
   - reader (already done previously with the **forcing translation**)
   - writer, exceptions, non-termination, non-determinism...
   - All with the new **weaning translation**!
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2. Implementing them thanks to program translations
   - No crazy category theory models!
   - So-called **syntactic models**.
   - Compile them on-the-fly into vanilla type theory!
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   - No crazy category theory models!
   - So-called **syntactic models**.
   - Compile them on-the-fly into vanilla type theory!

3. Introducing a generic notion of effectful dependent type theory
   - A simple, sensible restriction of dependent elimination
   - Seemingly compatible with all known effects
Syntactic Models

Define $[\cdot]$ on the syntax and derive the type interpretation $[\cdot]$ from it s.t.

$$\vdash M : A \quad \text{implies} \quad \vdash [M] : [A]$$
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$$\vdash M : A \quad \text{implies} \quad \vdash [M] : [[A]]$$

Obviously, that's subtle.

- The correctness of $[\cdot]$ lies in the meta (Darn, Gödel!)
- The translation must preserve typing (Not easy)
- In particular, it must preserve conversion (Argh!)
Syntactic Models

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Yet, a lot of nice consequences.

- Does not require non-type-theoretical foundations (monism)
- Can be implemented in your favourite proof assistant
- Easy to show (relative) consistency, look at $[\text{False}]$
- Easier to understand computationally
(Mis)understanding Dependent Type Theory

There are two essential properties of TT that need to be explicited.
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#1. Type theory is call-by-name by construction.

- This is because of the unrestricted conversion rule.
- But the usual monadic interpretation is call-by-value!
- We need to rely on an alternative decomposition (based on CBPV).
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**#1. Type theory is call-by-name by construction.**
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- We need to rely on an alternative decomposition (based on CBPV).

**#2. Dependent elimination is hardcore intuitionistic.**
- It rules out non-standard inductive terms that exist in CBN + effects
- Reminiscent of Brouwer vs. Bishop mathematics
- Needs to be weakened in presence of effects (« Bishop-style TT »)
TT is intrinsically call-by-name because of the conversion rule:

\[
\frac{\Gamma \vdash M : B \quad A \equiv_{\beta} B}{\Gamma \vdash M : A}
\]

where \(\equiv_{\beta}\) is generated by:

\[
(\lambda x : A. \ M) \ N \equiv_{\beta} M\{x := N\}
\]
My Name is Call, Call-by-Name

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where \( \equiv_{\beta} \) is generated by:

\[
(\lambda x : A. M) \ N \equiv_{\beta} M[x := N]
\]

To be call-by-value, it would require instead \( \equiv_{\beta v} \) generated by:

\[
(\lambda x : A. M) \ V \equiv_{\beta v} M[x := V]
\]

where \( V \) is a value. But that's not TT...
Tell Me Eleinberg-Moore

Turns out it is easy to give a call-by-name monadic decomposition.

Use the Eleinberg-Moore category, i.e. the category of algebras.
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**Use the Eleinberg-Moore category, i.e. the category of algebras.**

For us, a $T$-algebra will be an inhabitant of:

$$\blacksquare := \Sigma A : \blacksquare. \ T A \to A$$

A few remarks:

- It is hard to formulate the notion of algebra without higher-order types
- We don't require any equations in $\blacksquare$ (they're quite not algebras)
- It turns out it is not necessary...
Required structure

We assume a monad given by universe-polymorphic terms:

\[ T : \square_i \rightarrow \square_i \]
\[ \text{ret} : \Pi(A : \square). A \rightarrow T A \]
\[ \text{bind} : \Pi(A B : \square). T A \rightarrow (A \rightarrow T B) \rightarrow T B \]

and we require \textbf{no equations}!!
We assume a monad given by universe-polymorphic terms:

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\begin{align*}
T & : \Box_i \to \Box_i \\
\text{ret} & : \Pi(A : \Box). A \to T \, A \\
\text{bind} & : \Pi(A \, B : \Box). T \, A \to (A \to T \, B) \to T \, B
\end{align*}
\]

and we require **no equations**!!

Furthermore, in Type Theory, types are terms. We want the monad to be **self-algebraic**. This is given by:

\[
\begin{align*}
\text{El} & : T \, \Box_i \to \Box_i \\
\text{El} \, (\text{ret} \, \Box \, M) & \equiv_\beta M
\end{align*}
\]

A lot of monads appear to be self-algebraic.
The Weaning Translation of the Negative Fragment

\[
\begin{align*}
[x] & := x \\
[\lambda x : A. M] & := \lambda x : [A]. [M] \\
[M N] & := [M] [N] \\
[\Box_i] & := \text{ret} \Box_{i+1} (T \Box_i, \mu\Box) \\
[\Pi x : A. B] & := \text{ret} \Box (\Pi x : [A]. [B], \mu\Pi) \\
[A] & := (\text{El } [A]).\pi_1 \\
\mu\Box & : T (T \Box) \rightarrow \Box \\
\mu\Pi & : T (\Pi x : [A]. [B]) \rightarrow \Pi x : [A]. [B]
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[A] &:= (\text{El} \ [A]). \pi_1 \\
\mu_{\square} &:= T (T \ \square) \to \square \\
\mu_{\Pi} &:= T (\Pi x : [A]. [B]) \to \Pi x : [A]. [B]
\end{align*}
\]

- Functional fragment untouched, types mangled into algebras
- \([\square] \equiv_{\beta} T \ \square \) and \([\Pi x : A. B] \equiv_{\beta} \Pi x : [A]. [B] \)
The Weaning Translation of the Negative Fragment

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[\Pi x : A. \; B] & := \text{ret} \; \Box \; (\Pi x : [A]. [B], \mu_\Pi) \\
[A] & := (El \; [A]). \pi_1 \\
\mu_\Box & : T \; (T \; \Box) \rightarrow \Box \\
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- \([\Box] \equiv_\beta T \; \Box\) and \([\Pi x : A. \; B] \equiv_\beta \Pi x : [A]. [B]\)

Soundness

If \(\Gamma \vdash M : A\) then \([\Gamma] \vdash [M] : [A]\). (In particular, conversion is preserved.)
Reduction vs. Effects

Nothing fancy in the negative fragment, by the well-known duality.

- Call-by-name: **functions** well-behaved vs. **inductives** ill-behaved
- Call-by-value: **inductives** well-behaved vs. **functions** ill-behaved
Reduction vs. Effects

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- Call-by-value: **inductives** well-behaved vs. **functions** ill-behaved

Why is that?

In call-by-name + effects, consider:

\[(\lambda b : \text{bool}. \ M) \text{fail} \leadsto \text{non-standard inductive terms}\]

In call-by-value + effects, consider:

\[(\lambda b : \text{unit}. \text{fail}) \leadsto \text{invalid } \eta\text{-rule}\]
Weaning Inductive Types

For the sake of explanation, let's focus on a very simple type:

\[
\text{Inductive } \text{bool} := \text{true} | \text{false}.
\]

We pose:

\[
\begin{align*}
[\text{bool}] & := \text{ret } (T \text{bool}, \mu_{\text{bool}}) \\
[\text{true}] & := \text{ret bool true} \\
[\text{false}] & := \text{ret bool false} \\
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\end{align*}
\]

Remark that \([\text{bool}] \equiv_β T \text{ bool}\).

Soundness

If \(\Gamma ⊢ M : A\) then \([\Gamma] ⊢ [M] : [A]\).
We need a bit more structure on $T$ to implement elimination:

\[
\text{hbind} : \quad \Pi(A : \Box)(B : T \Box). \quad T \ A \to (A \to [B]) \to [B] \\
\text{dbind} : \quad \Pi(A : \Box)(B : A \to T \Box). \quad \Pi(\hat{x} : TA).
\quad (\Pi(x : A). [B \ x]) \to (\text{El} (\text{hbind} A [\ Box] \ \hat{x} B)). \pi_1
\]

subject to:

\[
\text{hbind} \ A \ B \ (\text{ret} \ A \ M) \ F \ \equiv_\beta \ F \ M \\
\text{dbind} \ A \ B \ (\text{ret} \ A \ M) \ F \ \equiv_\beta \ F \ M
\]

Essentially, hbind and dbind are variants of bind.
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Essentially, hbind and dbind are variants of bind.

Remark that the second equation is well-typed iff the first holds.
Interpreting Non-Dependent Elimination

It is easy to provide a non-dependent eliminator using `hbind`:

\[
\begin{array}{l}
[\text{bool\_case}] : [\prod P : □. P \to P \to \text{bool} \to P] \\
:= \lambda (P : T □)(p_t \, p_f : [P]) (\hat{b} : T \text{bool}). \\
\hspace{1em} \text{hbind bool } P \, \hat{b} \, (\lambda b. \text{if } b \text{ then } p_t \text{ else } p_f)
\end{array}
\]

which has the right reduction rules:

\[
\begin{array}{ll}
[\text{bool\_case } P \, p_t \, p_f \, \text{true}] \equiv_\beta p_t \\
[\text{bool\_case } P \, p_t \, p_f \, \text{false}] \equiv_\beta p_f
\end{array}
\]

Remember:

\[
\begin{array}{l}
\text{hbind : } \Pi (A : □)(B : T □). T \ A \to (A \to [B]) \to [B] \\
\text{hbind } A \ B \ (\text{ret } A \ M) \ F \equiv_\beta F \ M
\end{array}
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Eliminating Addiction to Dependence

We would like to recover dependent elimination...
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... but it's not valid anymore in presence of effects!

As $[\text{bool}] \equiv_{\beta} T \text{bool}$, if $T$ is not the identity then there are closed booleans in the translation which are neither $[\text{true}]$ nor $[\text{false}]$. 
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As $[\text{bool}] \equiv_{\beta} T \text{bool}$, if $T$ is not the identity then there are closed booleans in the translation which are neither $[\text{true}]$ nor $[\text{false}]$.

- Typical of CBN + effects: recall Herbelin's paradox
- Already arose in our forcing translation
- We need to restrict dependent elimination the same way!
The trick consists in sprinkling a few storage operators. For bool:

\[
\begin{align*}
[\theta_{\text{bool}}] & : \Gamma \rightarrow ([\text{bool} \rightarrow (\text{bool} \rightarrow □) \rightarrow □] \\
& := \lambda b. \text{bool}_\text{case} (\text{bool} \rightarrow □) (\lambda k. k \text{ true}) (\lambda k. k \text{ false}) b
\end{align*}
\]

- Only defined in the source via non-dependent eliminator
- In particular, agnostic to the actual translation
- CPS-like to enforce CBV in a CBN world
- Trivial in CIC: \( \vdash \Pi b : \text{bool}. \theta_{\text{bool}} b P = P b \)
The trick consists in sprinkling a few storage operators. For bool:

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[\theta_{\text{bool}}] : [\text{bool} \to (\text{bool} \to \Box) \to \Box] \\
:= [\lambda b. \text{bool\_case} (\text{bool} \to \Box) (\lambda k. k \text{ true}) (\lambda k. k \text{ false}) b]
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- Only defined in the source via non-dependent eliminator
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- CPS-like to enforce CBV in a CBN world
- Trivial in CIC: \[ \vdash \Pi b : \text{bool}. \theta_{\text{bool}} b P = P b \]

Using \text{dbind}, this allows to implement:

\[ [\text{bool\_rect}] : [\Pi P : \text{bool} \to \Box. P \text{ true} \to P \text{ false} \to \Pi b : \text{bool}. \theta_{\text{bool}} b P] \]

with the expected reduction rules.
Weaning Everywhere

There are a lot of monads that satisfy the weaning conditions.

- Exception monad $T A := A + E$
- Non-determinism $T A := A \times \text{list } A$
- Non-termination $T A := \nu X. A + X$
- Writer $T A := A \times \text{list } \Omega$ (the one we need for \texttt{HELLO WORLD})

Note that some lead to a logically inconsistent model.
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Note that some lead to a logically inconsistent model.

A few monads aren't self-algebraic, e.g. state, reader and continuation.
Logic, at Last

In some inconsistent cases, full dependent elimination is valid. Most notably, this is the case for the exception monad.

Let's use that to do a Friedman $A$-translation on steroids!
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Let's use that to do a Friedman $A$-translation on steroids!

Lemmas

With the exception monad $T A := A + E$:

- Full dependent elimination is valid (at the expense of consistency)
- We have $\lnot\lnot A \cong ([A] \rightarrow E) \rightarrow E$
- If $A$ is a first-order type, then $[A] \rightarrow A + E$. 
Logic, at Last

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Let's use that to do a Friedman $A$-translation on steroids!

**Lemmas**

With the exception monad $T \ A := A + E$:

- Full dependent elimination is valid (at the expense of consistency)
- We have $\llbracket \neg \neg A \rrbracket \cong (\llbracket A \rrbracket \rightarrow E) \rightarrow E$
- If $A$ is a first-order type, then $\llbracket A \rrbracket \rightarrow A + E$.

**Admissibility of Markov's rule in CIC**

If $A$ is first-order and $\vdash_{\text{CIC}} \neg \neg A$ then $\vdash_{\text{CIC}} A$. 

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Back to restricted elimination. It turns out we have a semantic criterion for valid dependent predicates.

**LINEARITY.**
Moi, j'ai dit linéaire, linéaire ? Comme c'est étrange...

Back to restricted elimination. It turns out we have a semantic criterion for valid dependent predicates.

**LINEARITY.**

- A concept invented by G. Munch, rephrased recently by P. Levy.
- Little to do with « linear use of variables »
- Essentially, $f: A \rightarrow B$ linear in CBN if semantically CBV in $A$.
- Categorically, $f$ linear iff it is an algebra morphism.
- Storage operators turn freely any morphism into a linear one.
- Can be approximated by a syntactic guard condition.

\[
\Gamma \vdash M : bool \quad \ldots \quad P \text{ linear in } b \\
\Gamma \vdash \text{if } M \text{ return } \lambda b. P \text{ then } N_1 \text{ else } N_2 : P\{b := M\}
\]
A Bishop-style Type Theory

We can generalize this restriction to form **Baclofen Type Theory**.

- Subset of CIC
- Independent from the actual translation.
- Works with forcing
- Works with weaning
- Prevents Herbelin's paradox
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**BTT is the generic theory to deal with dependent effects**

« Bishop-style, effect-agnostic type theory »

(Take that, Brouwerian HoTT!)
A nice paper summarizing this talk.

https://www.pédrot.fr/articles/weaning.pdf

Just as for the forcing translation we have a Coq plugin for weaning.

https://github.com/CoqHott/coq-effects

- Allows to add effects to Coq just today.
- Implement your favourite effectful operators: fail, fix...
- Compile effectful terms on the fly.
- Allows to reason about them in Coq.

(If time permits, small demo here.)
Conclusion

- A new effectful translation of TT, the weaning translation
  - Cosmic version of Eilenberg-Moore categories
  - Gives both programming and logical features

- An experimentally confirmed notion of effectful type theories, BTT
  - Works for forcing, weaning and CPS
  - Restriction of dependent elimination on linearity guard condition
  - Conjecture: the correct way to add effects to TT

- Implementation of a plugin in Coq
  - Try it out today!
Thanks for your attention.