A Parametric CPS to Sprinkle CIC with Classical Reasoning

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- You can program with it

  "A pure functional programming with crazily precise types."

- You can prove with it

  "A incredibly rich constructive logic with built-in computation."

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  "A incredibly rich constructive logic with built-in computation."
- Everything at the same time!

  "Prove your programs! Program your proofs!"

An effective object

That's just not theoretical ramble.

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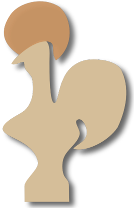

Lots of actual, serious, big developments.

- CompCert, VST, RustBelt...
- Four Colour Theorem, Feit-Thompson...

A Classical Problem

In practice, many people reason in the dreaded classical logic.

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Both a theoretical and practical limitation!

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- Most non-logicians don't care about this fuss (both CS and math...)

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It would be nice to have a $\mathfrak{classical}$ type theory...

Attempt 1: The Truth is Out There

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$$\texttt{Axiom classical : forall (A : Type), A} \lor \neg\texttt{A.}$$

Pro: Simple, local, works in Coq, be my guest.

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Pro: Simple, local, works in Coq, be my guest.

Cons:

- Axioms are dangerous, you have to show consistency externally

  Classical logic holds in the well-known **Set** *model, blah-blah...*
- Non-trivial interactions: e.g. classical CIC implies proof-irrelevance.

  Classical logic is incompatible with univalence! (Your mileage may vary.)
- The logic does not compute anymore, axioms block reduction...

Attempt 2: CIC and `call/cc` are in a boat

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$$\texttt{callcc} : ((A \to B) \to A) \to A$$

Essentially allows to reify context evaluation.

$$E[\texttt{callcc}\ M] \equiv_\beta \texttt{callcc}\ (\lambda k.\, E[M\ (E \circ k)])$$

The type of `callcc` is Peirce's law, the minimal logic equivalement of EM.

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Cons:

- Needs a whole new proof assistant implementation.

  Reminder: Coq is a 33-year old project.

- Changes the global meaning of logical connectives.

  What does $\Sigma x : A.\, B$ means?

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- … and it changes it so much that it also proves `False`!!!

  Pro: At least my proofs are going to be easier.

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Dependent elimination + Proof-relevance + `callcc` = TROUBLE.

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Dependent elimination + Proof-relevance + `callcc` = TROUBLE.

Essentially:

- `callcc` allows to build booleans that are neither `true` nor `false`

$$b := \texttt{if em CIC\_consistency then true else false}$$

- Dependent elimination is oblivious of this fact

$$\Pi P : \mathbb{B} \to \square.\, P\ \texttt{true} \to P\ \texttt{false} \to \Pi b : \mathbb{B}.\, P\ b$$

- Modern avatar of "Axiom of choice in classical logic is fishy".

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Come to see my LICS talk for a potential generic solution to CIC + effects!

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Buzzword: *linearity*. (Little to do with syntactic linearity BTW.)

$$\frac{\Gamma \vdash M : \mathbb{B} \qquad \Gamma \vdash N_1 : P\,\mathtt{true} \qquad \Gamma \vdash N_2 : P\,\mathtt{false} \qquad P \text{ linear in } b}{\Gamma \vdash \mathtt{if}\ M\ \mathtt{then}\ N_1\ \mathtt{else}\ N_2 : P\{b := M\}}$$

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- Works for CBN forcing
- Works for our new weaning translation
- Inspired by classical realizability
- Prevents Herbelin's particular paradox
- **Unluckily, a consistent model of callcc is still missing!**

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Observations:

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Continuation-passing style!

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We propose in this talk a much less grand solution than linearity.

The first *cheating* CPS translation of CIC.

Syntactic Models, a.k.a. Program Translations of CIC

Define $[\cdot]$ on the syntax and derive the type interpretation $[\![\cdot]\!]$ from it s.t.

$$\vdash_{\mathsf{CIC}^+} M : A \qquad \text{implies} \qquad \vdash_{\mathsf{CIC}} [M] : [\![A]\!]$$

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Obviously, that's subtle.

- The correctness of $[\cdot]$ lies in the meta (Darn, Gödel!)
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Yet, a lot of nice consequences.

- Does not require non-type-theoretical foundations (*monism*)
- Can be implemented in your favourite proof assistant
- Easy to show (relative) consistency, look at $[\![\texttt{False}]\!]$
- Easier to understand computationally

Baby steps

CIC is call-by-name by construction.

That's because of the β-equivalence used in conversion.

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We have to use a CBN CPS translation.

Let's stick to a variant close to the hardware: Lafont-Streicher-Reus CPS.

(This is LOLA after all.)

Quick recap

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2. Inductively define the type of stacks $\mathbb{C}(A)$ and witnesses $\mathbb{W}(A)$.

$$\begin{array}{lcl} \mathbb{W}(A) & := & \mathbb{C}(A) \rightarrow \perp\!\!\!\perp \\ \mathbb{C}(\alpha) & := & \alpha \rightarrow \perp\!\!\!\perp \\ \mathbb{C}(A \rightarrow B) & := & \mathbb{W}(A) \times \mathbb{C}(B) \end{array}$$

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3. Define the term translation $[\cdot]$ on the syntax s.t.

$$\Gamma \vdash M : A \quad \rightsquigarrow \quad \mathbb{W}(\Gamma) \vdash [M] : \mathbb{W}(A)$$

This Is LOLA After All

Here is the implementation:

$$\begin{array}{lcl} [x] & := & x \\ [\lambda x.\, M] & := & \lambda(x, \omega).\, [M]\ \omega \\ [M\ N] & := & \lambda \omega.\, [M]\ (N, \omega) \end{array}$$

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Holy celestial teapot! It implements the Krivine machine!

$$\begin{array}{lcl} \langle \lambda x.\, M \mid N \cdot \pi \rangle & \rightarrow & \langle M\{x := N\} \mid \pi \rangle \\ \langle M\ N \mid \pi \rangle & \rightarrow & \langle M \mid N \cdot \pi \rangle \end{array}$$

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Plus there is a proof of:

$$\mathbb{W}(((A \rightarrow B) \rightarrow A) \rightarrow A)$$

mimicking what the classical KAM does.

CICking it out

So far so good, we have a syntactic model for simply-typed λ-calculus.

Sketchy roadmap of what we have to do to scale LSR to CIC:

1. Acknowledging dependent functions
2. Implementing types-as-terms
3. Implementing dependent elimination

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1. Acknowledging dependent functions
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Spoiler: Turns out 1. is trivial, 2. and 3. impossible as-is.

LSR and dependency

Owing to the low-level nature of LSR, dependency is trivial.

$$
\begin{array}{lcl}
\mathbb{W}(A) & := & \mathbb{C}(A) \to \perp\!\!\!\perp \\
\mathbb{C}(A \to B) & := & \mathbb{W}(A) \times \mathbb{C}(B) \\
\mathbb{C}(\Pi x : A.\, B) & := & \Sigma x : \mathbb{W}(A).\, \mathbb{C}(B)
\end{array}
$$

Remark in particular that the arrow case is a degenerate variant.

It means it is easy to give a LSR of $\lambda\Pi$ s.t.

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Note: not as easy for other CBN CPS! So LSR is good for dependency.

LSR and inductive types

In LSR, inductive types are translated free algebras, e.g.

$$\begin{array}{lcl} \mathbb{C}(\mathbb{B}) & := & \mathbb{B} \to \perp\!\!\!\perp \\ \mathbb{W}(\mathbb{B}) & := & (\mathbb{B} \to \perp\!\!\!\perp) \to \perp\!\!\!\perp \end{array}$$

Constructors are returns, elimination is continuation-passing.

$$\begin{array}{lcl} [\mathtt{true}] & := & \lambda\omega.\,\omega\ \mathtt{true} \\ [\mathtt{false}] & := & \lambda\omega.\,\omega\ \mathtt{false} \\ [\mathtt{if}\ M\ \mathtt{then}\ N_1\ \mathtt{else}\ N_2] & := & \lambda\omega.\,[M]\ (\lambda b.\,\mathtt{if}\ b\ \mathtt{then}\ [N_1]\ \omega\ \mathtt{else}\ [N_2]\ \omega) \end{array}$$

LSR and inductive types: a failure

Alas, no hope to implement dependent elimination!

$$\Pi P : \mathbb{B} \to \square.\, P\ \mathtt{true} \to P\ \mathtt{false} \to \Pi b : \mathbb{B}.\, P\ b$$

$\rightsquigarrow$ For a meta-theoretical reason:

$\mathbb{W}(\mathbb{B}) := (\mathbb{B} \to \perp\!\!\!\perp) \to \perp\!\!\!\perp$, so depending on the choice of $\perp\!\!\!\perp$ there are non-standard booleans.

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In the typing of `if`, the type of a dependent ω would be wrong.

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No way to recover an actual boolean from a classical boolean.

LSR and universes: failure again

Because $\vdash_{\mathsf{CIC}} \square_i : \square_{i+1}$, we must define $\mathbb{C}(\square_i)$.

Universes are somehow free algebras, so take $\mathbb{C}(\square_i) := \square_i \to \perp\!\!\!\perp$.
In particular, $\mathbb{W}(\square_i) := (\square_i \to \perp\!\!\!\perp) \to \perp\!\!\!\perp$.

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In particular, $\mathbb{W}(\square_i) := (\square_i \to \perp\!\!\!\perp) \to \perp\!\!\!\perp$.

Now, how to implement the meta-function $\mathrm{El} : \mathbb{W}(\square) \rightsquigarrow \square$, needed for

$$\frac{\Gamma \vdash A : \square_i}{\vdash \Gamma, A : \square_i}$$

Actually, you can't. Just as for booleans, double-negation lost information.

No way to recover an actual type from a classical type either.

A Dire Situation

TL; DR: LSR handles negative connectives but not positive ones.

Not totally unexpected from a CPS translation...

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Let's make the CPS intuitionistic again by using..

Parametricity.

Or equivalently, let's do a bit of...

Intuitionistic realizability.

The Grand Scheme

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Idea: instead of translating

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let's rather do

$$\Gamma \vdash M : A \quad \rightsquigarrow \quad [\![\Gamma]\!] \vdash [M]^{!} : [\![A]\!]$$

where

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We will retrieve the information in $\cdot \in A$ rather than in $\mathbb{W}(A)$!

$M \in A$ is the parametricity (resp. realizability) relation of A.

The Grand Scheme II

Morally, our translation is

- Intuitionistic Realizability (Kleene-style?)
- ... where realizers are Lafont-Streicher-Reus CPS-ified terms
- ... and where the realizability relation is internal to CIC

A fancy mix... Is that a known technique?

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- ... and where the realizability relation is internal to CIC

A fancy mix... Is that a known technique?

Has it a use per se? Can it be used for type-preserving compilation?

A Bit of Detail

Compared from the simply-typed case, $[\cdot]$ is unchanged.

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We define the realizability condition as follows:

A	$\mathbb{C}(A)$	$(M : \mathbb{C}(A) \to \perp\!\!\!\perp) \in A$
$\Pi x : A.\, B$	$\Sigma x : [\![A]\!].\, \mathbb{C}(B)$	$\Pi x : [\![A]\!].\, (\lambda \omega.\, M\ (x, \omega)) \in B$

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$\square$	$\square \rightarrow \perp\!\!\!\perp$	$\Sigma X : \square.\, (M = \texttt{ret}\ X) \times ((M \rightarrow \perp\!\!\!\perp) \rightarrow \square)$

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$\mathbb{B}$	$\mathbb{B} \to \bot\!\!\!\bot$	$\Sigma b : \mathbb{B}. M = \texttt{ret}\ b$
$\square$	$\square \to \bot\!\!\!\bot$	$\Sigma X : \square.\, (M = \texttt{ret}\ X) \times ((M \to \bot\!\!\!\bot) \to \square)$

Technically, $[\![A]\!]$, $\mathbb{C}(A)$ and $M \in A$ are macros derived from $[A]_\varepsilon$.

A Few Isomorphims

This translation is very intuitionistic, as it is somehow the identity.

Assuming $\perp\!\!\!\perp$ is hProp:

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In particular, it preserves consistency!

The only difference (due to parametricity):

$$[\![\square]\!] \ncong \square$$

Soundness

Interestingly, this translation can be carried in CIC.

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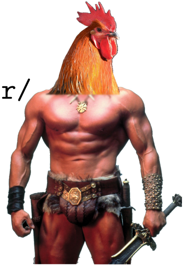
If $\Gamma \vdash_{\mathsf{CIC}} M : A$ then $[\![\Gamma]\!] \vdash_{\mathsf{CIC}} [M]^! : [\![A]\!]$

So it is possible to provide this translation as Coq plugin!
For now only a hand-written shallow embedding.
`https://github.com/CoqHott/coq-effects/blob/master/theories/misc/CPS.v`



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What did we gain? Not a lot of things...

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- ... in particular it **negates** univalence!

That said, we have new statements in our theory.

Sprinkling Classical Logic

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Behold the classical modality $\langle\cdot\rangle$!

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Sprinkling Classical Logic

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The modality just drops the parametric proof of the underlying type.

$$[\![\langle A\rangle]\!] := \Sigma x : \mathbb{W}(A).\,\texttt{unit} \cong \mathbb{W}(A)$$

As such, it allows to work with the raw LSR translation.

Moar Principles

This type constructor admits a lot of reasoning principles.

- It has a return:

$$\eta : \Pi(A : \Box).\, A \to \langle A \rangle$$

- It has (a weak form of) choice:

$$\Pi(x : A).\, \langle B \rangle \cong \langle \Pi(x : A).\, B \rangle$$

- It has a form of classical reasoning:

$$\mathsf{cc} : \Pi(A\ B : \Box).\, ((A \to \langle B \rangle) \to \langle A \rangle) \to \langle A \rangle$$

- It is **not** functorial.

$$A \to B \not\vdash \langle A \rangle \to \langle B \rangle$$

- In particular, it is **not** the double negation modality.

Give Me My Propositional Logic Back

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Furthermore, the propositional logic combinators compute. E.g.

$$\mathtt{if}^{\langle\cdot\rangle} : \langle \mathbb{B} \rangle \to \langle A \rangle \to \langle A \rangle \to \langle A \rangle$$

$$\mathtt{if}^{\langle\cdot\rangle}\ (\eta\ \mathbb{B}\ \mathtt{true})\ N_1\ N_2 \equiv_\beta N_1$$

This is all because the LSR CPS is well-behaved w.r.t. β-reduction.

Obviously no dependent elimination in sight. (Because LSR.)

Even More

For particular values of $\perp\!\!\!\perp$, we get more. Typically, for $\perp\!\!\!\perp := \texttt{empty}$.

- The modality is consistent.

$$\langle \texttt{empty} \rangle \rightarrow \texttt{empty}$$

- The modality has excluded middle.

$$\texttt{em} : \Pi(A : \square).\, \langle A + \neg A \rangle$$

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- The first typed CPS of CIC!
- Although we cheat badly.
- An intricate mix of techniques.
- Implementable in Coq.
- A modality $\langle \cdot \rangle$ introducing classical logic.
- Preserving the propositional fragment, not dependent elimination.

Again, what can we do with that?

Thanks for your attention.