Une Théorie des Types qui fait de l'effet

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CIC: « Constructions dans un monde qui bouge »

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- First class notion of computation and crazy inductive types

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The Pinnacle of the Curry-Howard correspondence

An Effective Object

One implementation to rule them all...

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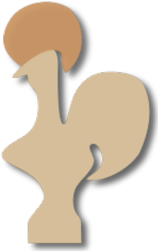

An Effective Object

One implementation to rule them all...

Many big developments using it for computer-checked proofs.

- Mathematics: Four colour theorem, Feit-Thompson, Unimath...
- Computer Science: CompCert, VST, RustBelt...

What are dependent types?

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Trivial : *types depend on programs*

$$\texttt{hd} : \Pi(A : \texttt{Type})\,(n : \mathbb{N}).\,\texttt{vect}\ A\ (n + 1) \rightarrow A$$

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$$\mathtt{hd} : \Pi(A : \mathtt{Type})\,(n : \mathbb{N}).\,\mathtt{vect}\ A\ (n+1) \to A$$

Important : n is quantified over **terms** of the language.

(*Chassez cette méta-théorie que je ne saurais voir !*)

Less trivial: *types depend on computation*

let

$$\begin{array}{lcl} \texttt{init} & : & \Pi n : \mathbb{N}.\, \texttt{vect}\ \mathbb{N}\ n \\ \texttt{init}\ n & := & [0; \ldots; n-1] \end{array}$$

and

$$\begin{array}{lcl} g & : & \texttt{option}\ \mathbb{N} \to \mathbb{N} \\ g\ \texttt{None} & := & 0 \\ g\ (\texttt{Some}\ n) & := & n+1 \end{array}$$

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because

$$g\ (\mathtt{Some}\ 42) \quad \equiv \quad 43$$

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It computes everywhere!

WTF CIC?

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MER IL ET FOU

You will have a little more *mind-blowing*?

Proofs are *relevant*.

"Any list can be quotiented by permutations."

- Quicksort
- Mergesort
- Bogosort

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Proofs are first-class objects.

What about $\Pi(A : \mathtt{Type}).\, \Pi(x : A).\, \Pi(p\, q : x = x).\, p = q$?

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Un indice, chez vous

This innocent-looking question dragged a Fields medalist into type theory.

The dependent product : the heroin of type theory

The dependent product, a generalization of arrow types?

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$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x : A.\, t : \Pi x : A.\, B}$$

$$\frac{\Gamma \vdash t : \Pi x : A.\, B \qquad \Gamma \vdash u : A}{\Gamma \vdash t\ u : B\{x := u\}}$$

$$(\lambda x : A.\, t)\, u \quad \equiv \quad t\{x := u\}$$

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$$(\lambda x : A.\, t)\, u \quad \equiv \quad t\{x := u\}$$

Answer: Yes.

$$A \to B \quad \equiv \quad \Pi(\_ : A).\, B$$

Dependent elimination: mind the withdrawal

The other *killer feature* of dependent types.

$$\frac{\Gamma \vdash t_1 : A}{\Gamma \vdash \texttt{inl}\ t_1 : A + B} \qquad \frac{\Gamma \vdash t_2 : B}{\Gamma \vdash \texttt{inr}\ t_2 : A + B}$$

$$\frac{\Gamma, s : A + B \vdash P : \texttt{Type} \quad \Gamma \vdash p_1 : \Pi(x : A).\, P\{\texttt{inl}\ x\} \quad \Gamma \vdash p_2 : \Pi(y : B).\, P\{\texttt{inr}\ y\}}{\Gamma \vdash \texttt{rec}_+\ P\ p_1\ p_2 : \Pi(s : A + B).\, P}$$

$$\texttt{rec}_+\ P\ p_1\ p_2\ (\texttt{inl}\ t_1) \equiv p_1\ t_1$$
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Just a beefed up pattern-matching!

$$\texttt{rec}_+\ P\ p_1\ p_2\ s \equiv \texttt{match}\ s\ \texttt{with}\ \texttt{inl}\ x \Rightarrow p_1\ x \mid \texttt{inr}\ y \Rightarrow p_2\ y$$

The type of a branch can change depending on it.

Inductive types

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Can even be generalized to fancy types, e.g. equality.

$$\frac{\Gamma \vdash A : \texttt{Type} \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : B}{\Gamma \vdash \texttt{eq}\ A\ t\ u : \texttt{Type}}$$

$$\frac{\Gamma \vdash A : \texttt{Type} \qquad \Gamma \vdash t : A}{\Gamma \vdash \texttt{refl}\ A\ t : \texttt{eq}\ A\ t\ t}$$

$$\frac{\Gamma, y : A, e : \texttt{eq}\ A\ t\ y \vdash P : \texttt{Type} \qquad \Gamma \vdash p : P\{t, \texttt{refl}\ A\ t\}}{\Gamma \vdash \texttt{rec}_{\texttt{eq}}\ P\ p : \Pi(y : A)\,(e : \texttt{eq}\ A\ t\ y).\, P}$$

$$\texttt{rec}_{\texttt{eq}}\ P\ p\ t\ (\texttt{refl}\ A\ t) \equiv p$$

This implements everything expected from equality!

Wololo

Conversion, the formal way to express that types depend on computation.

$$\frac{\Gamma \vdash M : B \qquad \Gamma \vdash A \equiv B}{\Gamma \vdash M : A}$$

Dallas, ton univers impitoyable

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$$\Pi(A : \texttt{Type}).\, P \quad \sim \quad \texttt{'a. P} \text{ in OCaml}$$

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(For conciseness we will often write $\square := \texttt{Type}$.)

Summary

We can think of CIC as follows.

- A Haskell-like functional language
- $+$ Π generalizing arrows
- $+$ ADTs with generalized pattern-matching
- $+$ inclusion of types in terms
- Proof and computation living in harmony

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"The earthly paradise of the proof-program correspondence."

“The earthly paradise”?

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Moi, jamais je pipeaute.

The CIC brothers

Actually not quite one single theory.

Several flags tweaking the kernel:

- Impredicative Set
- Type-in-type
- Indices Matter
- Cumulative inductive types
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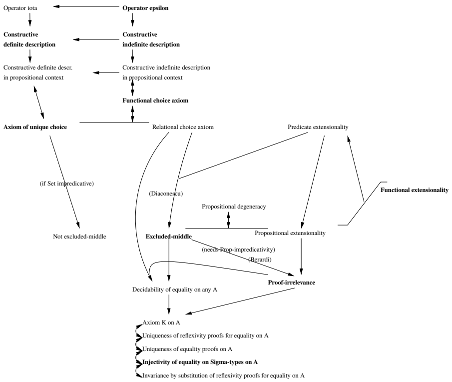

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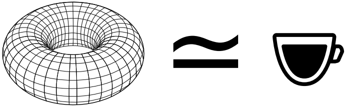

« A mathematician is a device for turning toruses into equalities (up to homotopy). »

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Varying degrees of compatibility.

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Mathematicians may not care too much, but...

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COULD YOU WRITE A HELLO WORLD PROGRAM PLEASE?

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Intuitionistic Logic $\Leftrightarrow$ **Functional** Programming

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Non-Intuitionistic Logic $\Leftrightarrow$ **Impure** Programming

- `callcc` gives classical logic
- Delimited continuations prove Markov's principle
- Exceptions implement Markov's rule
- (Certain) presheaves provide univalence
- ...

We want a type theory with **effects** !

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We want a model of type theory with effects.

1. The theory ought to be logically consistent
2. It should be implementable (e.g. decidable type-checking)
3. Other nice properties like canonicity ($\vdash n : \mathbb{N}$ implies $n \rightsquigarrow \mathtt{S} \ldots \mathtt{S}\,\mathtt{0}$)

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Effectful models of CIC will be discussed on Friday.

(Diabolical laughter.)

The Ancestral Path

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- Requires a new proof of soundness (... *cough*... right, F*? *cough*...)
- Implementation task may be daunting (including bugs)
- Yet-another-language: say farewell to libraries, tools, community...

Summary of the Problem

Different users have different needs.

« From each according to his ability, to each according to his needs. »

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Are we thus doomed?

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One *backend* implementation to rule them all!

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One **backend** implementation to rule them all!

via

Syntactic Models

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Roughly three families of models:

- The **set-theoretical** model and its variants
- Several **realizability** models
- A gazillion of **categorical** models

Let's review them quickly!

The Set-Theoretical Model

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Interpret everything as sets and expect $\vdash_{\mathsf{CIC}} M : A \Rightarrow \vdash_{\mathsf{ZFC}} [M] \in [A]$.

$$[\Pi x : A.\, B] \equiv \left\{ f \in [A] \rightarrow_{\mathsf{ZFC}} \bigcup_{x \in [A]} [B](x) \;\middle|\; \forall x \in [A].\, f(x) \in [B](x) \right\}$$

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- Forego syntax, computation and decidability
- No effects in sight.
- Imports ZFC properties.

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Construct programs that respect properties.

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- Terms $M \rightsquigarrow$ programs $[M]$ (variable languages as a target)
- Types $A \rightsquigarrow$ meta-theoretical predicates $[\![A]\!]$
- $\vdash_{\mathsf{CIC}} M : A \quad \Rightarrow \quad [M] \in [\![A]\!]$

$$[\![\Pi x : A.\, B]\!] \equiv \{f \in \Lambda \mid \forall x \in [\![A]\!].\, \texttt{eval}(f, x) \in [\![B]\!](x)\}$$

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- Usually crazily undecidable
- Meta-theory can be arbitrary crap, including ZFC

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Con

- Same limitations as the previous examples
- Mostly useless to actually construct a model
- Yet another syntax, usually arcane and ill-fitted

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For instance, in $\mathbf{Set}$:

$$[\mathrm{Prop}] \equiv \{\varnothing, \{\varnothing\}\}$$

so in there you can inhabit e.g.

$$\texttt{prop\_ext} : \Pi(A\ B : \mathrm{Prop}).\,(A \leftrightarrow B) \rightarrow A = B$$

$$\texttt{em} : \Pi(A : \mathrm{Prop}).\,A + \neg A$$

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Five seconds of thorough thinking for the sleepy ones.

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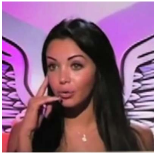
- Interprets it into some low-level language.
- Must preserve the meaning of the source.
- Refines the behaviour of under-specified structures.

Five seconds of thorough thinking for the sleepy ones.

« Non mais allô quoi, this is a *compiler*... »

You are a logician and you don't know Curry-Howard?



Curry-Howard Orthodoxy

Let's look at what Curry-Howard provides in simpler settings.

Program Translations $\Leftrightarrow$ Logical Interpretations

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- Monadic style à la Haskell
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Program Translations $\Leftrightarrow$ Logical Interpretations

On the **programming** side, enrich the language by program translation.

- Monadic style à la Haskell
- Compilation of higher-level constructs down to assembly

On the **logic** side, extend expressivity through proof interpretation.

- Double-negation $\Rightarrow$ classical logic (`callcc`)
- Friedman's trick $\Rightarrow$ Markov's rule (exceptions)
- Forcing $\Rightarrow$ $\neg$CH (global monotonous cell)

Syntactic Models

Let us do the same thing with CIC: build **syntactic models**.

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We take the following act of faith for granted.

CIC is.

Syntactic Models

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We take the following act of faith for granted.

CIC is.

Not caring for its soundness, implementation, whatever. It just is.

Do everything by interpreting the new theories relatively to this foundation!

Suppress technical and cognitive burden by lowering impedance mismatch.

Syntactic Models II

Step 0: Fix a theory $\mathcal{T}$ as close as possible* to CIC, ideally CIC $\subseteq \mathcal{T}$.

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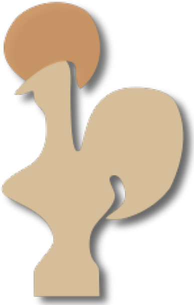
$$\vdash_{\mathcal{T}} M : A \qquad \stackrel{\Delta}{=} \qquad \vdash_{\mathsf{CIC}} [M] : [\![A]\!]$$

Step 3: Expand $\mathcal{T}$ by going down to the *CIC assembly language*, implementing new terms given by the $[\cdot]$ translation.

Anatomy of a syntactic model

COMPILATION



$\vdash_{\mathsf{CIC}^{++}} M : A \qquad \rightsquigarrow \qquad \vdash_{\mathsf{CIC}} [M] : [\![A]\!]$

« CIC, the LLVM of type theory »

Syntactic Models III

Obviously, that's subtle.

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- In particular, it must preserve conversion (even worse)

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Yet, a lot of nice consequences.

- Does not require non-type-theoretical foundations (*monism*)
- **Can be implemented in Coq** (*software monism*)
- Easy to show (relative) consistency, look at $[\![\texttt{False}]\!]$
- Inherit properties from CIC: computationality, decidability...

A Few Examples

In the remainder, I'll focus on simple examples of syntactic models.

- Mostly pedagogical
- In particular, no effects involved (?)
- Still funny to mess with CIC

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Wait for Friday for me to talk about an effectful CIC.

Models of the day

1. Intensional functions
2. Intensional types, and their utmost horrifying realization
3. Parametricity

Intensional Functions

(A very simple introductory example.)

On Extensionality

Ever heard about *function extensionality*?

$$\texttt{funext} : \Pi(A : \texttt{Type})\,(B : A \to \texttt{Type})(f\,g : \Pi(x : A).\,B\;x).$$
$$(\Pi(x : A).\,f\,x = g\;x) \to f = g$$

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Turns out it is not provable in CIC.

... even though most sane people add it as an **axiom**.

Negating Functional Extensionality

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Let's take advantage of this by mangling functions in our model.

- Namely, attach a boolean to them
- Will not affect β-reduction
- Will be observable by intensional equality

Technical details

$$
\begin{array}{lcl}
[x] & := & x \\
[\lambda x : A.\, M] & := & (\lambda x : [\![A]\!].\, [M], \mathtt{true}) \\
[M\, N] & := & [M].\pi_1\, [N] \\
[\Box] & := & \Box \\
[\Pi x : A.\, B] & := & (\Pi x : [\![A]\!].\, [\![B]\!]) \times \mathtt{bool} \\
[\ldots] & := & \ldots \\
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$$

(Inductive types mostly untouched.)

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(Inductive types mostly untouched.)

Soundness

We have $\Gamma \vdash M : A$ implies $[\![\Gamma]\!] \vdash [M] : [\![A]\!]$.

Negating Functional Extensionality II

This means that you can extend the source theory with

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda' x : A.\, M : \Pi x : A.\, B}$$

defined as:

$$[\lambda' x : A.\, M] := (\lambda x : [\![A]\!].\, [M], \texttt{false})$$

Rembember:

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Clearly this new abstraction has the same behaviour as the original one.

$$[(\lambda' x : A.\, M)\, N] \equiv [M\{x := N\}]$$

Negating Functional Extensionality III

Now, it is easy to see how to negate functional extensionality. Consider:

$$\Sigma(f\,g : \mathbb{1} \to \mathbb{1}).\,(\Pi i : \mathbb{1}.\, f\ i = g\ i) \land f \neq g$$

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(The actual translation is a little noisier, but this does not change the idea.)

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(The actual translation is a little noisier, but this does not change the idea.)

Take $f := [\lambda x : \mathbb{1}.\, x]$ and $g := [\lambda' x : \mathbb{1}.\, x]$, and *voilá*!

Where We Cheated

We did not explicit the rules of the source theory.

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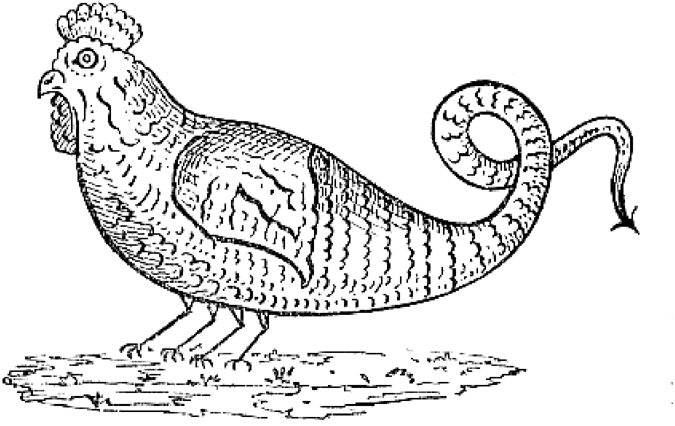
In particular, it is clear that the model invalidates η-rules.

$$\begin{array}{ccc} [\lambda x : A.\, M\ x] & \not\equiv & [M] \\ \equiv & & \equiv \\ (\lambda x : [\![A]\!].\, [M].\pi_1\ x, \mathtt{true}) & \not\equiv & [M] \end{array}$$

It's much harder to negate extensionality while preserving η.
(We'll see than on Friday.)



Intensional Types, a.k.a. **Dynamically Typed CIC**

Intensional Types

The intensional types translation extends type theory with

$$\begin{array}{lcl} \texttt{flip} & : & \square \to \square \\ \texttt{flip\_equiv} & : & \Pi(A : \square).\,\texttt{flip}\ A \cong A \\ \texttt{flip\_neq} & : & \Pi(A : \square).\,\texttt{flip}\ A \neq A \end{array}$$

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This breaks amongst other things univalence...

The Intensional Types Implementation

Intuitively:

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Soundness

If $\vec{x} : \Gamma \vdash M : A$ then $\vec{x} : [\![\Gamma]\!] \vdash [M] : [\![A]\!]$.

Extending the Intensional Types

Let's define the new operations obtained through the translation.

$$
\begin{array}{lcl}
[\mathtt{flip}] & : & [\![\square \to \square]\!] \\
[\mathtt{flip}] & : & \square \times \mathbb{B} \to \square \times \mathbb{B} \\
[\mathtt{flip}] & \equiv & \lambda(A, b).\, (A, \mathtt{negb}\ b) \\
\\
[\mathtt{flip\_equiv}] & : & [\![\Pi A : \square.\, \mathtt{flip}\ A \cong A]\!] \\
[\mathtt{flip\_equiv}] & \equiv & \ldots \\
\\
[\mathtt{flip\_neq}] & : & [\![\Pi A : \square.\, \mathtt{flip}\ A \neq A]\!] \\
[\mathtt{flip\_neq}] & : & \Pi A : \square \times \mathbb{B}.\, [\mathtt{flip}]\ A \neq A \\
[\mathtt{flip\_equiv}] & \equiv & \ldots
\end{array}
$$

- $[\![\mathtt{flip}\ A]\!] \equiv [\![A]\!]$
- And isomorphism only depends on $[\![A]\!]$
- But (intensional) equality observes the boolean...

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- Assuming the target theory features induction-recursion
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$$
\begin{array}{ll}
\texttt{type\_rect}: & \Pi(P : \square \to \square). \\
& \quad P\ \square \to \\
& \quad (\Pi(A : \square)\,(B : A \to \square).\ P\ A \to (\Pi x : A.\ P\ (B\ x)) \to \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad P\ (\Pi x : A.\ B)) \to \\
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Coq is compatible with dynamic types!!!

Parametricity

A Brief Recap on Parametricity

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Usually phrased as a set-theoretic relation between terms.

$$\text{If } \vdash_{\mathsf{F}} t : A \text{ then } (t, t) \in [\![A]\!]$$

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Instead of System F

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General idea:

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- Variables $x : A$ in the context are triplicated into x_0, x_1, x_ε

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[\Gamma, x : A]_\varepsilon & \equiv & [\Gamma]_\varepsilon, x_0 : [A]_0, x_1 : [A]_1, x_\varepsilon : [A]_\varepsilon\ x_0\ x_1
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$$

If $\Gamma \vdash M : A$ then $[\Gamma]_\varepsilon \vdash [M]_i : [A]_i$
$[\Gamma]_\varepsilon \vdash [M]_\varepsilon : [A]_\varepsilon\ [M]_0\ [M]_1.$

Inductive Parametricity

Translation of inductive types is just as simple (but not easy).

- $\mathcal{I} : \square$ is mapped to $\mathcal{I}_\varepsilon : \mathcal{I}_0 \to \mathcal{I}_1 \to \square$
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\quad (B : \square) : \\
\quad \square := \\
\mid \texttt{inl} : \Pi(x : A). \\
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\quad (B_0\, B_1 : \square)\, (B_\varepsilon : B_0 \to B_1 \to \square) : \\
\quad \texttt{sum}\ A_0\ B_0 \to \texttt{sum}\ A_1\ B_1 \to \square := \\
\mid \texttt{inl}_\varepsilon : \Pi(x_0 : A_0)\, (x_1 : A_1)\, (x_\varepsilon : A_\varepsilon\ x_0\ x_1). \\
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\mid \texttt{inl} : \Pi(x : A). \\
\qquad \texttt{sum}\ A\ B \\
\mid \texttt{inr} : \Pi(y : B). \\
\qquad \texttt{sum}\ A\ B
\end{array}
$$

$$
\begin{array}{l}
\texttt{Inductive sum}_\varepsilon \\
\quad (A_0\, A_1 : \square)\,(A_\varepsilon : A_0 \to A_1 \to \square) \\
\quad (B_0\, B_1 : \square)\,(B_\varepsilon : B_0 \to B_1 \to \square) : \\
\quad \texttt{sum}\ A_0\ B_0 \to \texttt{sum}\ A_1\ B_1 \to \square := \\
\mid \texttt{inl}_\varepsilon : \Pi(x_0 : A_0)\,(x_1 : A_1)\,(x_\varepsilon : A_\varepsilon\ x_0\ x_1). \\
\qquad \texttt{sum}_\varepsilon\ A_0\ A_1\ A_\varepsilon\ B_0\ B_1\ B_\varepsilon\ (\texttt{inl}\ x_0)\ (\texttt{inl}\ x_1) \\
\mid \texttt{inr}_\varepsilon : \Pi(y_0 : B_0)\,(y_1 : B_1)\,(y_\varepsilon : B_\varepsilon\ y_0\ y_1). \\
\qquad \texttt{sum}_\varepsilon\ A_0\ A_1\ A_\varepsilon\ B_0\ B_1\ B_\varepsilon\ (\texttt{inr}\ y_0)\ (\texttt{inr}\ y_1)
\end{array}
$$

Parametricity interprets all of CIC!

Parametricity as a Model

Parametricity is as a slightly more complex kind of syntactic model.

Instead of only **one** component, we have **three**.

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We define the theory CIC_p as $\Gamma \vdash_{\mathsf{CIC}_p} M : A$ whenever:

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Clearly CIC $\subseteq$ CIC$_p$ by soundness.

What about the additional expressive power of CIC$_p$?

Patatras!

CIC$_p$ is a **conservative extension** of CIC.

That is, if $\Gamma, M, A \in \mathsf{CIC}$ and $\Gamma \vdash_{\mathsf{CIC}_p} M : A$ then $\Gamma \vdash_{\mathsf{CIC}} M : A$.

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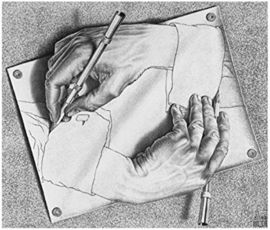
Parametricity is just repeating properties that were already there.



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Non-classicality

We have $\not\vdash_{\mathsf{CIC}_p} \Pi(A : \square).\, A + \neg A$.

Corollary

We have $\not\vdash_{\mathsf{CIC}} \Pi(A : \square).\, A + \neg A$.

The Power of Computers

It's extremely tedious to write parametricity proofs by hand.

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Thankfully, we said that this model was a compiler!

Just implement it as a Coq plugin!

- This was done by Keller and Lasson
- Allows to systematically generate horrible proof-terms
- Computer science rules

`https://github.com/coq-community/paramcoq`

Midterm Conclusion

- Syntactic models are a handy tool
- Easier to comprehend (I think)
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- Implement them for Coq as mere plugins

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Be there on Friday to enter the frightening realm of effects.

Scribitur ad narrandum, non ad probandum.

Merci de votre attention.