Une Théorie des Types qui fait de l’effet

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CIC, the Calculus of Inductive Constructions.
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CIC, a very fancy intuitionistic logical system.

- Not just higher-order logic, not just first-order logic
- First class notion of computation and crazy inductive types
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CIC, a very powerful **functional programming language**.
- Finest types to describe your programs
- No clear phase separation between runtime and compile time
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The Pinnacle of the Curry-Howard correspondence
An Effective Object

One implementation to rule them all...
An Effective Object

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Many big developments using it for computer-checked proofs.

- Mathematics: Four colour theorem, Feit-Thompson, Unimath...
- Computer Science: CompCert, VST, RustBelt...
What are dependent types?

Trivial: types depend on programs

$$
\text{hd} : \prod(A : \text{Type}) \, (n : \mathbb{N}) \rightarrow \text{vect}_A(n + 1) \rightarrow A
$$

Important: $n$ is quantified over terms of the language.

(Chassez cette méta-théorie que je ne saurais voir !)
What are dependent types?

Trivial: *types depend on programs*

\[ \text{hd} : \Pi (A : \text{Type}) (n : \mathbb{N}). \text{vect } A (n + 1) \to A \]
What are dependent types?

Trivial: types depend on programs

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Important: \( n \) is quantified over terms of the language.

\((\text{Chassez cette méta-théorie que je ne saurais voir!})\)
Slightly non-trivial

Less trivial: *types depend on computation*

let

\[
\text{init} : \Pi n : \mathbb{N}. \text{vect} \mathbb{N} n
\]

\[
\text{init} \ n \ := \ [0; \ldots; n - 1]
\]

and

\[
g : \text{option} \mathbb{N} \rightarrow \mathbb{N}
\]

\[
g \ \text{None} \ := \ 0
\]

\[
g \ (\text{Some} \ n) \ := \ n + 1
\]
Slightly non-trivial

Less trivial: 

\[ \text{let} \]
\[
\begin{align*}
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\[ \text{and} \]
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g & : \text{option} \mathbb{N} \rightarrow \mathbb{N} \\
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g (\text{Some} \ n) & := n + 1
\end{align*}
\]

\[ \text{then} \]
\[
\text{init} (g (\text{Some} \ 42)) : \text{vect} \mathbb{N} \ 43
\]

\[ \text{because} \]
\[
g (\text{Some} \ 42) \equiv 43
\]
Slightly non-trivial

Less trivial: *types depend on computation*

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then

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\text{init } (g (\text{Some } 42)) : \text{vect } \mathbb{N} \ 43
\]

because

\[
g (\text{Some } 42) \equiv 43
\]

It computes everywhere!
What is going on: Ça dépend, ça dépasse.

\[ \lambda b: \mathbb{B}. \text{if } b \text{ then } 0 \text{ else true} \]
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\[
\lambda b : \mathbb{B}. \text{if } b \text{ then } 0 \text{ else true } : \Pi b : \mathbb{B}. \text{if } b \text{ then } \mathbb{N} \text{ else } \mathbb{B}
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\]

MER IL ET FOU
You will have a little more *mind-blowing*?

**Proofs are relevant.**

| “Any list can be quotiented by permutations.” | • Quicksort  
|                                            | • Mergesort  
|                                            | • Bogosort  

Un indice, chez vous

This innocent-looking question dragged a Fields medalist into type theory.

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You will have a little more *mind-blowing*?

Proofs are *relevant*.

“Any list can be quotiented by permutations.”

- Quicksort
- Mergesort
- Bogosort

Proofs are first-class objects.

What about \( \Pi(A : \text{Type}). \Pi(x : A). \Pi(p \ q : x = x). p = q \)?
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What about \( \Pi(A : \text{Type}). \Pi(x : A). \Pi(p q : x = x). p = q \)?

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The dependent product, a generalization of arrow types?

\[
\Gamma \vdash A : \text{Type} \quad \Gamma, x : A \vdash B : \text{Type} \\
\Gamma \vdash \Pi x : A. B : \text{Type}
\]
The dependent product: the heroin of type theory

The dependent product, a generalization of arrow types?

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\Gamma \vdash A : \text{Type} \quad \Gamma, x : A \vdash B : \text{Type} \\
\Gamma \vdash \Pi x : A. B : \text{Type}
\]

\[
\Gamma, x : A \vdash t : B \\
\Gamma \vdash \lambda x : A. t : \Pi x : A. B
\]

\[
\Gamma \vdash t : \Pi x : A. B \quad \Gamma \vdash u : A \\
\Gamma \vdash t u : B\{x := u\}
\]

\[
(\lambda x : A. t) u \equiv t\{x := u\}
\]
The dependent product: the heroin of type theory

The dependent product, a generalization of arrow types?

\[
\begin{align*}
\Gamma \vdash A : Type \quad \Gamma, x : A \vdash B : Type \quad & \quad \Gamma \vdash \Pi x : A. B : Type \\
\Gamma, x : A \vdash t : B \quad & \quad \Gamma \vdash \lambda x : A. t : \Pi x : A. B \\
\Gamma \vdash t : \Pi x : A. B \quad \Gamma \vdash u : A \quad & \quad \Gamma \vdash t\ u : B\{x := u\} \\
(\lambda x : A. t)\ u \quad \equiv \quad t\{x := u\}
\end{align*}
\]

Answer: Yes.

\[A \rightarrow B \quad \equiv \quad \Pi(\_ : A). B\]
Dependent elimination: mind the withdrawal

The other *killer feature* of dependent types.

\[
\Gamma \vdash t_1 : A \\
\Gamma \vdash \text{inl } t_1 : A + B \\
\Gamma \vdash t_2 : B \\
\Gamma \vdash \text{inr } t_2 : A + B
\]

\[
\Gamma, s : A + B \vdash P : \text{Type} \\
\Gamma \vdash p_1 : \Pi(x : A). P\{\text{inl } x\} \\
\Gamma \vdash p_2 : \Pi(y : B). P\{\text{inr } y\}
\]

\[
\Gamma \vdash \text{rec}_+ \; P \; p_1 \; p_2 : \Pi(s : A + B). P
\]

\[
\text{rec}_+ \; P \; p_1 \; p_2 \; (\text{inl } t_1) \equiv p_1 \; t_1
\]

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\Gamma \vdash \text{rec}_+ \; P \; p_1 \; p_2 : \Pi(s : A + B). \; P \\
\text{rec}_+ \; P \; p_1 \; p_2 \; (\text{inl} \; t_1) & \equiv p_1 \; t_1 \\
\text{rec}_+ \; P \; p_1 \; p_2 \; (\text{inr} \; t_2) & \equiv p_2 \; t_2
\end{align*}
\]

Just a beefed up pattern-matching!

\[
\text{rec}_+ \; P \; p_1 \; p_2 \; s \equiv \text{match} \; s \; \text{with} \; \text{inl} \; x \Rightarrow p_1 \; x \; | \; \text{inr} \; y \Rightarrow p_2 \; y
\]

The type of a branch can change depending on it.
Logically, dependent elimination corresponds to induction principles.
Inductive types

Logically, dependent elimination corresponds to induction principles.

Can even be generalized to fancy types, e.g. equality.

\[
\begin{align*}
\Gamma \vdash A : \text{Type} & \quad \Gamma \vdash t : A \quad \Gamma \vdash u : B \\
\hline \\
\Gamma \vdash \text{eq } A \ t \ u : \text{Type} \\
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash A : \text{Type} & \quad \Gamma \vdash t : A \\
\hline \\
\Gamma \vdash \text{refl } A \ t : \text{eq } A \ t \ t \\
\end{align*}
\]

\[
\begin{align*}
\Gamma, y : A, e : \text{eq } A \ t \ y & \vdash P : \text{Type} \quad \Gamma \vdash p : P\{t, \text{refl } A \ t\} \\
\hline \\
\Gamma \vdash \text{rec}_{\text{eq}} P \ p : \Pi(y : A) (e : \text{eq } A \ t \ y). P \\
\end{align*}
\]

\[
\text{rec}_{\text{eq}} P \ p \ t (\text{refl } A \ t) \equiv p
\]

This implements everything expected from equality!
Conversion, the formal way to express that types depend on computation.

\[
\Gamma \vdash M : B \quad \Rightarrow \quad \Gamma \vdash A \equiv B
\]

\[
\Gamma \vdash M : A
\]
Without entering details too much, types are also terms.

\[ \Pi(A : \text{Type}). P \sim 'a. \ P \text{ in OCaml} \]
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For instance, we wrote:

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In particular, \( \text{Type} : \text{Type} \).

(Almost.)
Without entering details too much, types are also terms.

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For instance, we wrote:

\[
\frac{\Gamma \vdash A : \text{Type} \quad \Gamma, x : A \vdash B : \text{Type}}{\Gamma \vdash \Pi x : A. B : \text{Type}}
\]

In particular, Type : Type.

(Almost.)

(For conciseness we will often write \( \Box := \text{Type} \).)
We can think of CIC as follows.

- A Haskell-like functional language
- $\Pi$ generalizing arrows
- ADTs with generalized pattern-matching
- Inclusion of types in terms
- Proof and computation living in harmony
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“The earthly paradise of the proof-program correspondence.”
“The earthly paradise”? 
“The earthly paradise”? 

Moi, jamais je pipeaute.
Actually not quite one single theory.

Several flags tweaking the kernel:
- Impredicative Set
- Type-in-type
- Indices Matter
- Cumulative inductive types
- ...

The Many Calculi of Inductive Constructions.
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The CIC brothers

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In the Axiom Jungle

A crazy amount of axioms used in the wild!
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The classical set-theory pole:
- Excluded middle, UIP, choice
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The univalent pole:
- Univalence, what else?

« A mathematician is a device for turning toruses into equalities (up to homotopy). »
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The \textbf{exotic} pole:
  - Anti-classical axioms (???)
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Varying degrees of compatibility.
Theorem 0

Axioms Suck.
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Proof.

- They break computation (and thus canonicity).
- They are hard to justify.
- They might be incompatible with one another.
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Axioms Suck.

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- They break computation (and thus canonicity).
- They are hard to justify.
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Mathematicians may not care too much, but...
The Most Important Issue of Them All

CIC suffers from an even more fundamental flaw.
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CIC suffers from an even more **fundamental** flaw.

- You want to show the wonders of Coq to a fellow programmer
- You fire your favourite IDE
- ... and you’re asked the **DREADFUL** question.
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- … and you’re asked the **DREADFUL** question.

**COULD YOU WRITE A HELLO WORLD PROGRAM PLEASE?**
Sad reality

Intuitionistic Logic ⇔ Functional Programming

Coq is even purer than Haskell:
- No mutable state (obviously)
- No exceptions (Haskell has them somehow)
- No arbitrary recursion
- and also no Hello World!

We want a type theory with effects!
Sad reality

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We want a type theory with **effects**!
The Good News

**Intuitionistic Logic ⇔ Functional Programming**
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is by folklore the same as

Non-Intuitionistic Logic ⇔ Impure Programming
The Good News

Intuitionistic Logic ⇔ Functional Programming

is by folklore the same as

Non-Intuitionistic Logic ⇔ Impure Programming

- callcc gives classical logic
- Delimited continuations prove Markov’s principle
- Exceptions implement Markov’s rule
- (Certain) presheaves provide univalence
- ...

We want a type theory with effects!
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1. To program more (exceptions, non-termination...)
2. To prove more (classical logic, univalence...)
3. To write Hello World.

It's not just randomly coming up with typing rules though. We want a model of type theory with effects.

1. The theory ought to be logically consistent
2. It should be implementable (e.g. decidable type-checking)
3. Other nice properties like canonicity ($\vdash n : \mathbb{N} \Rightarrow n \Rightarrow S \ldots S \Omega$)

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For the remainder of this talk, we will concentrate on the notion of models.
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Effectful models of CIC will be discussed on Friday.

(Diabolical laughter.)
Let us assume we defined formally what a model was.
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Assuming we have a model, we can **implement** a new type theory.

Examples: Cubical, F*...
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**Pro**

- Computational by construction (hopefully)
- Tailored for a specific theory
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Examples: Cubical, F*...

**Pro**
- Computational by construction (hopefully)
- Tailored for a specific theory

**Con**
- Requires a new proof of soundness (... cough... right, F*? cough...)
- Implementation task may be daunting (including bugs)
- Yet-another-language: say farewell to libraries, tools, community...
Summary of the Problem

Different users have different needs.

« From each according to his ability, to each according to his needs. »
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« From each according to his ability, to each according to his needs. »

(Excessive) Fragmentation of proof assistants is harmful.

« Divide et impera. »
Summary of the Problem

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« From each according to his ability, to each according to his needs. »

(Excessive) Fragmentation of proof assistants is harmful.

« Divide et impera. »

Are we thus doomed?
In this talk, I’d like to advocate for a way out of the conundrum.

One implementation to rule them all...
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One backend implementation to rule them all!

via

Syntactic Models
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I won’t lie: they are. But part of this fame is due to its usual models.
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Roughly three families of models:

- The \textit{set-theoretical} model and its variants
- Several \textit{realizability} models
- A gazillion of \textit{categorical} models

Let’s review them quickly!
The Set-Theoretical Model

Because Sets are a type theory.
The Set-Theoretical Model

Because Sets are a type theory.

Interpret everything as sets and expect $\vdash_{\text{CIC}} M : A \Rightarrow \vdash_{\text{ZFC}} [M] \in [A]$.

$$\lfloor \Pi x : A. B \rfloor \equiv \left\{ f \in [A] \rightarrow_{\text{ZFC}} \bigcup_{x \in [A]} [B](x) \mid \forall x \in [A]. f(x) \in [B](x) \right\}$$
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- Well-known and trusted target
- Imports ZFC properties.
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Pro

- Well-known and trusted target
- Imports ZFC properties.

Con

- Forego syntax, computation and decidability
- No effects in sight.
- Imports ZFC properties.
The Realizability Models

Construct programs that respect properties.
The Realizability Models

Construct programs that respect properties.

- Terms $M \rightsquigarrow$ programs $[M]$ (variable languages as a target)
- Types $A \rightsquigarrow$ meta-theoretical predicates $[A]
- \vdash_{\text{CIC}} M : A \implies [M] \in [A]

$$[\Pi x : A. B] \equiv \{ f \in \Lambda \mid \forall x \in [A]. \text{eval}(f, x) \in [B](x) \}$$
The Realizability Models

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**Pro**

- Some preservation of syntax and computability

**Con**

- Usually crazily undecidable
- Meta-theory can be arbitrary crap, including ZFC

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The Categorical Models

Abstract description of type theory.
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Rephrase the rules of CIC in a categorical way.
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- Very abstract and subsumes both previous examples
- Somewhat “easier” to show some structure is a model of TT
The Categorical Models

Abstract description of type theory.

Rephrase the rules of CIC in a categorical way.

Pro
- Very abstract and subsumes both previous examples
- Somewhat “easier” to show some structure is a model of TT

Con
- Same limitations as the previous examples
- Mostly useless to actually construct a model
- Yet another syntax, usually arcane and ill-fitted
Assuming we pick a specific model, what do we do with it?
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Hopefully it has a more refined content!

In particular, you can show that an axiom hold in this model.
What’s The Matter

Assuming we pick a specific model, what do we do with it?

Hopefully it has a more refined content!

In particular, you can show that an axiom hold in this model.

For instance, in Set:

\[
[\text{Prop}] \equiv \{\emptyset, \{\emptyset\}\}
\]

so in there you can inhabit e.g.

\[
\text{prop\_ext} : \Pi(A \, B : \text{Prop}). (A \leftrightarrow B) \rightarrow A = B
\]

\[
\text{em} : \Pi(A : \text{Prop}). A + \neg A
\]
What is a model?

- Takes syntax as input.
- Interprets it into some low-level language.
- Must preserve the meaning of the source.
- Refines the behaviour of under-specified structures.
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Five seconds of thorough thinking for the sleepy ones.
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Five seconds of thorough thinking for the sleepy ones.

« Non mais allô quoi, this is a **compiler**... »

*You are a logician and you don’t know Curry-Howard?*
Let’s look at what Curry-Howard provides in simpler settings.

Program Translations $\iff$ Logical Interpretations
Curry-Howard Orthodoxy

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Program Translations $\Leftrightarrow$ Logical Interpretations

On the **programming** side, enrich the language by program translation.
- Monadic style à la Haskell
- Compilation of higher-level constructs down to assembly
Curry-Howard Orthodoxy

Let’s look at what Curry-Howard provides in simpler settings.

Program Translations $\Leftrightarrow$ Logical Interpretations

On the **programming** side, enrich the language by program translation.
- Monadic style à la Haskell
- Compilation of higher-level constructs down to assembly

On the **logic** side, extend expressivity through proof interpretation.
- Double-negation $\Rightarrow$ classical logic (callcc)
- Friedman’s trick $\Rightarrow$ Markov’s rule (exceptions)
- Forcing $\Rightarrow$ $\neg$CH (global monotonous cell)
Let us do the same thing with CIC: build **syntactic models**.
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We take the following act of faith for granted.

**CIC is.**
Let us do the same thing with CIC: build **syntactic models**.

We take the following act of faith for granted.

**CIC is.**

Not caring for its soundness, implementation, whatever. It just is.

Do everything by interpreting the new theories relatively to this foundation!

Suppress technical and cognitive burden by lowering impedance mismatch.
Step 0: Fix a theory $\mathcal{T}$ as close as possible* to CIC, ideally $\text{CIC} \subseteq \mathcal{T}$.
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Step 1: Define $[\cdot]$ on the syntax of $\mathcal{T}$ and derive $[\cdot]$ from it s.t.

$$\vdash_{\mathcal{T}} M : A \quad \text{implies} \quad \vdash_{\text{CIC}} [M] : [A]$$
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Step 1: Define $[\cdot]$ on the syntax of $\mathcal{T}$ and derive $[[\cdot]]$ from it s.t.

$$\vdash_{\mathcal{T}} M : A \quad \text{implies} \quad \vdash_{\text{CIC}} [M] : [[A]]$$

Step 2: Flip views and actually pose

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Step 0: Fix a theory $\mathcal{T}$ as close as possible* to CIC, ideally $\text{CIC} \subseteq \mathcal{T}$.

Step 1: Define $\llbracket \cdot \rrbracket$ on the syntax of $\mathcal{T}$ and derive $\llbracket \cdot \rrbracket$ from it s.t.

$$\vdash_{\mathcal{T}} M : A \quad \text{implies} \quad \vdash_{\text{CIC}} \llbracket M \rrbracket : \llbracket A \rrbracket$$

Step 2: Flip views and actually pose

$$\vdash_{\mathcal{T}} M : A \quad \triangleq \quad \vdash_{\text{CIC}} \llbracket M \rrbracket : \llbracket A \rrbracket$$

Step 3: Expand $\mathcal{T}$ by going down to the CIC assembly language, implementing new terms given by the $\llbracket \cdot \rrbracket$ translation.
Anatomy of a syntactic model

\[ \vdash_{\text{CIC++}} M : A \leadsto \vdash_{\text{CIC}} [M] : [A] \]

"CIC, the LLVM of type theory"
Obviously, that’s subtle.

- The translation $[\cdot]$ must preserve typing (not easy)
- In particular, it must preserve conversion (even worse)
Obviously, that’s subtle.

- The translation $\cdot$ must preserve typing (not easy)
- In particular, it must preserve conversion (even worse)

Yet, a lot of nice consequences.

- Does not require non-type-theoretical foundations (*monism*)
- **Can be implemented in Coq** (*software monism*)
- Easy to show (relative) consistency, look at $[\text{False}]$
- Inherit properties from CIC: computationality, decidability...
In the remainder, I’ll focus on simple examples of syntactic models.

- Mostly pedagogical
- In particular, no effects involved (?)
- Still funny to mess with CIC
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- Mostly pedagogical
- In particular, no effects involved (?)
- Still funny to mess with CIC

Wait for Friday for me to talk about an effectful CIC.
Models of the day

1. Intensional functions
2. Intensional types, and their utmost horrifying realization
3. Parametricity
Intensional Functions

(A very simple introductory example.)
Ever heard about *function extensionality*?

\[
\text{funext} : \prod(A : \text{Type}) (B : A \to \text{Type}) (f, g : \prod(x : A). B x).
\]

\[
(\prod(x : A). f x = g x) \to f = g
\]
Ever heard about *function extensionality*?

\[
\text{funext} : \Pi(A : \text{Type})(B : A \to \text{Type})(f \ g : \Pi(x : A). B x).
\]

\[
(\Pi(x : A). f \ x = g \ x) \to f = g
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Turns out it is not provable in CIC.
Ever heard about function extensionality?

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\]
\[
(\prod(x : A). f \ x = g \ x) \to f = g
\]

Turns out it is not provable in CIC.

... even though most sane people add it as an axiom.
If it is not provable, let’s break it!
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The only thing you know about functions:

\[(\lambda x : A. M) \equiv M\{x := N\}\]
Negating Functional Extensionality

If it is not provable, let’s break it!

The only thing you know about functions:

$$(\lambda x : A. M) N \equiv M\{x := N\}$$

Let’s take advantage of this by mangling functions in our model.

- Namely, attach a boolean to them
- Will not affect $\beta$-reduction
- Will be observable by intensional equality
Technical details

\[
\begin{align*}
[x] & := x \\
[\lambda x : A. M] & := (\lambda x : [A]. [M], \text{true}) \\
[M N] & := [M]. \pi_1 [N] \\
[\square] & := \square \\
[\Pi x : A. B] & := (\Pi x : [A]. [B]) \times \text{bool} \\
[\ldots] & := \ldots \\
[A] & := [A]
\end{align*}
\]

(Inductive types mostly untouched.)
Technical details

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\end{align*}
\]

(Inductive types mostly untouched.)

Soundness

We have $\Gamma \vdash M : A$ implies $[\Gamma] \vdash [M] : [A]$. 
Negating Functional Extensionality II

This means that you can extend the source theory with

\[
\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda'x : A. M : \Pi x : A. B}
\]

defined as:

\[
[\lambda'x : A. M] := (\lambda x : [A]. [M], false)
\]

Remembember:

\[
\begin{align*}
[\lambda x : A. M] & := (\lambda x : [A]. [M], true) \\
[M N] & := [M].\pi_1 [N]
\end{align*}
\]
Negating Functional Extensionality II

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\Gamma, x : A \vdash M : B \\
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\]

defined as:

\[
[\lambda' x : A. M] := (\lambda x : [A]. [M], \text{false})
\]

Rembember:

\[
[\lambda x : A. M] := (\lambda x : [A]. [M], \text{true}) \\
[M \, N] := [M].\pi_1 [N]
\]

Clearly this new abstraction has the same behaviour as the original one.

\[
[(\lambda' x : A. M) \, N] \equiv [M\{x := N\}]
\]
Negating Functional Extensionality III

Now, it is easy to see how to negate functional extensionality. Consider:

$$\Sigma(f \circ g : \mathbb{1} \rightarrow \mathbb{1}). (\Pi i : \mathbb{1}. f \, i = g \, i) \land f \neq g$$
Now, it is easy to see how to negate functional extensionality. Consider:

\[ \Sigma(fg : 1 \rightarrow 1). (\Pi i : 1. f i = g i) \land f \neq g \]

This is translated into something that is essentially:

\[ \Sigma(fg : (1 \rightarrow 1) \times \text{bool}). (\Pi i : 1. f.\pi_1 i = g.\pi_1 i) \land f \neq g \]

(The actual translation is a little noisier, but this does not change the idea.)
Now, it is easy to see how to negate functional extensionality. Consider:

$$
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$$

(The actual translation is a little noisier, but this does not change the idea.)

Take $f := \lambda x : 1. x$ and $g := \lambda' x : 1. x$, and voilá!
Where We Cheated

We did not explicit the rules of the source theory.
Where We Cheated

We did not explicit the rules of the source theory.

In particular, it is clear that the model invalidates $\eta$-rules.

$$[\lambda x : A. \ M \ x] \not\equiv [M]$$

$$\lambda x : [A]. [M]. \pi_1 \ x, \text{true} \not\equiv [M]$$

It’s much harder to negate extensionality while preserving $\eta$.
(We’ll see than on Friday.)
Intensional Types, a.k.a. **Dynamically Typed CIC**
The intensional types translation extends type theory with

\[
\begin{align*}
\text{flip} & : \square \to \square \\
\text{flip}_\text{equiv} & : \Pi(A : \square). \text{flip } A \simeq A \\
\text{flip}_\text{neq} & : \Pi(A : \square). \text{flip } A \neq A
\end{align*}
\]
The intensional types translation extends type theory with

\[
\begin{align*}
\text{flip} & : \Box \to \Box \\
\text{flip\_equiv} & : \Pi(A : \Box). \text{flip } A \cong A \\
\text{flip\_neq} & : \Pi(A : \Box). \text{flip } A \neq A
\end{align*}
\]

This breaks amongst other things univalence...
The Intensional Types Implementation

Intuitively:

- Translate $A : \Box$ into $[A] : \Box \times \mathbb{B}$
- Translate $M : A$ into $[M] : [A].\pi_1$
The Intensional Types Implementation

Intuitively:

- Translate $A : \square$ into $[A] : \square \times \mathbb{B}$
- Translate $M : A$ into $[M] : [A].\pi_1$

\[
\begin{align*}
[A] & \equiv [A].\pi_1 \\
[\square] & \equiv (\square \times \mathbb{B}, \text{true}) \\
[\Pi x : A. B] & \equiv (\Pi x : [A].[B], \text{true}) \\
x & \equiv x \\
[M \ N] & \equiv [M] [N] \\
[\lambda x : A. M] & \equiv \lambda x : [A].[M]
\end{align*}
\]

Types contain a boolean not used for their inhabitants!
The Intensional Types Implementation

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[x] & \equiv x \\
[M \, N] & \equiv [M] \, [N] \\
[\lambda x : A. M] & \equiv \lambda x : [A].[M]
\end{align*}
\]

Types contain a boolean not used for their inhabitants!

Soundness

If $\bar{x} : \Gamma \vdash M : A$ then $\bar{x} : [\Gamma] \vdash [M] : [A]$. 
Let’s define the new operations obtained through the translation.

\[
\begin{align*}
\text{[flip]} & : \square \rightarrow \square \\
\text{[flip]} & : \square \times \mathbb{B} \rightarrow \square \times \mathbb{B} \\
\text{[flip]} & \equiv \lambda (A, b). (A, \neg b)
\end{align*}
\]

\[
\begin{align*}
\text{[flip\_equiv]} & : \Pi A : \square. \text{flip } A \cong A \\
\text{[flip\_equiv]} & \equiv \ldots
\end{align*}
\]

\[
\begin{align*}
\text{[flip\_neq]} & : \Pi A : \square. \text{flip } A \neq A \\
\text{[flip\_neq]} & : \Pi A : \square \times \mathbb{B}. \text{[flip]} A \neq A \\
\text{[flip\_equiv]} & \equiv \ldots
\end{align*}
\]

- \([\text{flip } A] \equiv [A]\)
- And isomorphism only depends on \([A]\)
- But (intensional) equality observes the boolean...
This one example is not very interesting.
Basilisk for Realz

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You can do much better: a real mix of Python and Coq!
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You can do much better: a real mix of Python and Coq!

- Assuming the target theory features induction-recursion
- Represent (source) types by their code
- This gives a real type-quote function in the source theory

```
    type_rect : \Pi (P : □ \to □).
                P □ \to
                (\Pi (A : □) (B : A \to □). P A \to (\Pi x : A. P (B x)) \to
                 P (\Pi x : A. B)) \to
                P \mathbb{N} \to
                \ldots \to
                \Pi (A : □). P A
```
Basilisk for Realz

This one example is not very interesting.

You can do much better: a real mix of Python and Coq!

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\[
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\]
\[
P \Box \to
\]
\[
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\]
\[
P \mathbb{N} \to
\]
\[
\ldots \to
\]
\[
\Pi(A : \Box). P A
\]

Coq is compatible with dynamic types!!!
Parametricity
You probably already have heard of **parametricity**.
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Originally invented by Reynolds to show properties for System F programs.

**Theorems for Free!**
A Brief Recap on Parametricity

You probably already have heard of parametricity.

Originally invented by Reynolds to show properties for System F programs.

Theorems for Free!

Usually phrased as a set-theoretic relation between terms.

\[
\text{If } \vdash_F t : A \text{ then } (t, t) \in \llbracket A \rrbracket
\]
Instead of System F

What if we used type theory as the source theory for parametricity?
A Type-Theoretic Parametricity

Instead of System F
What if we used type theory as the **source** theory for parametricity?

Instead of ZFC
What if we used type theory as the **target** theory for parametricity?

---

### Variables $x : A$ in the context are triplicated into $x_0$, $x_1$, $x \in Pédrot (Gallinette) Une Théorie des Types qui fait de l’effet JFLA 2019 53 / 61
A Type-Theoretic Parametricity

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That’s **exactly** what Bernardy-Lasson syntactic translation is about!

**General idea:**

- $A : \Box$ is mapped to $[A]_0 : \Box$, $[A]_1 : \Box$ and $[A]_\varepsilon : [A]_0 \to [A]_1 \to \Box$
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General idea:

- \( A : □ \) is mapped to \([A]_0 : □, [A]_1 : □ \) and \([A]_ε : [A]_0 \rightarrow [A]_1 \rightarrow □\)
- \( M : A \) is mapped to \([M]_i : [A]_i \) and \([M]_ε : [A]_ε [M]_0 [M]_1\)
A Type-Theoretic Parametricity

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General idea:

- $A : \Box$ is mapped to $[A]_0 : \Box$, $[A]_1 : \Box$ and $[A]_\varepsilon : [A]_0 \rightarrow [A]_1 \rightarrow \Box$
- $M : A$ is mapped to $[M]_i : [A]_i$ and $[M]_\varepsilon : [A]_\varepsilon [M]_0 [M]_1$
- Variables $x : A$ in the context are triplicated into $x_0, x_1, x_\varepsilon$
The Syntactic Translation

\[
\left[ M \right]_i \equiv M\{\bar{x} := \bar{x}_i\}
\]
The Syntactic Translation

\[ [M]_i \equiv M\{\vec{x} := \vec{x}_i\} \]

\[ [\Box]_\varepsilon \equiv \lambda(A_0A_1 : \Box). A_0 \to A_1 \to \Box \]

\[ [\Pi x : A. B]_\varepsilon \equiv \lambda(f_0 : [\Pi x : A. B]_0) (f_1 : [\Pi x : A. B]_1). \]

\[ \Pi(x_0 : [A]_0)(x_1 : [A]_1)(x_\varepsilon : [A]_\varepsilon x_0 x_1). \]

\[ [B]_\varepsilon (f_0 x_0)(f_1 x_1) \]
The Syntactic Translation

\[ [M]_i \equiv M\{\bar{x} := \bar{x}_i}\]

\[ [\square]_\varepsilon \equiv \lambda(A_0A_1 : \square). A_0 \rightarrow A_1 \rightarrow \square \]

\[ [\Pi x : A. B]_\varepsilon \equiv \lambda(f_0 : [\Pi x : A. B]_0) (f_1 : [\Pi x : A. B]_1).
\]

\[ \Pi(x_0 : [A]_0) (x_1 : [A]_1) (x_\varepsilon : [A]_\varepsilon x_0 x_1). \]

\[ [B]_\varepsilon (f_0 x_0) (f_1 x_1) \]

\[ [x]_\varepsilon \equiv x_\varepsilon \]

\[ [M N]_\varepsilon \equiv [M]_\varepsilon [N]_0 [N]_1 [N]_\varepsilon \]

\[ [\lambda x : A. M]_\varepsilon \equiv \lambda(x_0 : [A]_0) (x_1 : [A]_1) (x_\varepsilon : [A]_\varepsilon x_0 x_1). [M]_\varepsilon \]
The Syntactic Translation

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\quad \Pi(x_0 : [A]_0)(x_1 : [A]_1)(x_\varepsilon : [A]_\varepsilon x_0 x_1).
\quad [B]_\varepsilon(f_0 x_0)(f_1 x_1)

\[[x]_\varepsilon\] \equiv x_\varepsilon

\[[M N]_\varepsilon\] \equiv [M]_\varepsilon [N]_0 [N]_1 [N]_\varepsilon

\[[\lambda x : A. M]_\varepsilon\] \equiv \lambda(x_0 : [A]_0)(x_1 : [A]_1)(x_\varepsilon : [A]_\varepsilon x_0 x_1). [M]_\varepsilon

\[[\cdot]_\varepsilon\] \equiv \cdot

\[[\Gamma, x : A]_\varepsilon\] \equiv [\Gamma]_\varepsilon, x_0 : [A]_0, x_1 : [A]_1, x_\varepsilon : [A]_\varepsilon x_0 x_1

If \(\Gamma \vdash M : A\) then
\[[\Gamma]_\varepsilon \vdash [M]_i : [A]_i\]
\[[\Gamma]_\varepsilon \vdash [M]_\varepsilon : [A]_\varepsilon [M]_0 [M]_1.\]
Inductive Parametricity

Translation of inductive types is just as simple (but not easy).

- $\mathcal{I} : \square$ is mapped to $\mathcal{I}_\varepsilon : \mathcal{I}_0 \to \mathcal{I}_1 \to \square$
- Constructors are translated pointwise
- Dependent elimination is straightforward
Inductive Parametricity

Translation of inductive types is just as simple (but not easy).

- \( I : \square \) is mapped to \( I_\varepsilon : I_0 \to I_1 \to \square \)
- Constructors are translated pointwise
- Dependent elimination is straightforward

\[
\text{Inductive sum} \\
(A : \square) \\
(B : \square): \\
\square := \\
| \text{inl} : \Pi(x : A). \sum A B \\
| \text{inr} : \Pi(y : B). \sum A B
\]
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\[\text{Inductive sum} \quad (A : \square) \quad (B : \square) : \square := \]
\[
\begin{align*}
| \text{inl} & : \Pi(x : A). & \text{sum } A B \hfill \\
| \text{inr} & : \Pi(y : B). & \text{sum } A B \\
\end{align*}
\]

\[\text{Inductive sum}_\varepsilon \quad (A_0 A_1 : \square)(A_\varepsilon : A_0 \to A_1 \to \square) \quad (B_0 B_1 : \square)(B_\varepsilon : B_0 \to B_1 \to \square) : \sum A_0 B_0 \to \sum A_1 B_1 \to \square := \]
\[
\begin{align*}
| \text{inl}_\varepsilon & : \Pi(x_0 : A_0)(x_1 : A_1)(x_\varepsilon : A_\varepsilon \ x_0 \ x_1). & \sum_\varepsilon A_0 A_1 A_\varepsilon B_0 B_1 B_\varepsilon (\text{inl } x_0) (\text{inl } x_1) \\
| \text{inr}_\varepsilon & : \Pi(y_0 : B_0)(y_1 : B_1)(y_\varepsilon : B_\varepsilon \ y_0 \ y_1). & \sum_\varepsilon A_0 A_1 A_\varepsilon B_0 B_1 B_\varepsilon (\text{inr } y_0) (\text{inr } y_1) \\
\end{align*}
\]
Inductive Parametricity

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|
| Inductive sum $\varepsilon$
| $(A_0 A_1 : \Box)(A_\varepsilon : A_0 \to A_1 \to \Box)$
| $(B_0 B_1 : \Box)(B_\varepsilon : B_0 \to B_1 \to \Box)$:
| $\text{sum}_\varepsilon A_0 B_0 \to \text{sum}_\varepsilon A_1 B_1 \to \Box :=$
| $\text{inl}_\varepsilon : \Pi(x_0 : A_0)(x_1 : A_1)(x_\varepsilon : A_\varepsilon) x_0 x_1).$
| $\text{sum}_\varepsilon A_0 A_1 A_\varepsilon B_0 B_1 B_\varepsilon (\text{inl} x_0) (\text{inl} x_1)$
| $\text{inr}_\varepsilon : \Pi(y_0 : B_0)(y_1 : B_1)(y_\varepsilon : B_\varepsilon) y_0 y_1).$
| $\text{sum}_\varepsilon A_0 A_1 A_\varepsilon B_0 B_1 B_\varepsilon (\text{inr} y_0) (\text{inr} y_1)$

Parametricity interprets all of CIC!
Parametricity as a Model

Parametricity is as a slightly more complex kind of syntactic model.

Instead of only one component, we have three.
Parametricity as a Model

Parametricity is as a slightly more complex kind of syntactic model.

Instead of only one component, we have three.

**CIC\(_p\)**

We define the theory CIC\(_p\) as \(\Gamma \vdash_{\text{CIC}\(_p\)} M : A\) whenever:

- \([\Gamma]_0 \vdash_{\text{CIC}} [M]_0 : [A]_0\)
- \([\Gamma]_1 \vdash_{\text{CIC}} [M]_1 : [A]_1\)
- \([\Gamma]_\varepsilon \vdash_{\text{CIC}} [M]_\varepsilon : [A]_\varepsilon [M]_0 [M]_1\)
Parametricity as a Model

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Instead of only one component, we have three.

\[ \text{We define the theory } \text{CIC}_p \text{ as } \Gamma \vdash_{\text{CIC}_p} M : A \text{ whenever:} \]

1. \( [\Gamma]_0 \vdash_{\text{CIC}} [M]_0 : [A]_0 \)
2. \( [\Gamma]_1 \vdash_{\text{CIC}} [M]_1 : [A]_1 \)
3. \( [\Gamma]_\varepsilon \vdash_{\text{CIC}} [M]_\varepsilon : [A]_\varepsilon [M]_0 [M]_1 \)

Clearly \( \text{CIC} \subseteq \text{CIC}_p \) by soundness.
Parametricity as a Model

Parametricity is as a slightly more complex kind of syntactic model.

Instead of only **one** component, we have **three**.

<table>
<thead>
<tr>
<th>CIC&lt;sub&gt;p&lt;/sub&gt;</th>
</tr>
</thead>
</table>

We define the theory \( \text{CIC}_p \) as \( \Gamma \vdash_{\text{CIC}_p} M : A \) whenever:

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- \( [\Gamma]_\varepsilon \vdash_{\text{CIC}} [M]_\varepsilon : [A]_\varepsilon [M]_0 [M]_1 \)

Clearly \( \text{CIC} \subseteq \text{CIC}_p \) by soundness.

**What about the additional expressive power of CIC<sub>p</sub>?**
CIC\textsubscript{p} is a **conservative extension** of CIC.

That is, if $\Gamma, M, A \in \text{CIC}$ and $\Gamma \vdash_{\text{CIC}_p} M : A$ then $\Gamma \vdash_{\text{CIC}} M : A$. 

We did not get any expressivity over the CIC common subsystem. It is trivial to see: just pick the first (or second) projection. Parametricity is just repeating properties that were already there.
\[ \text{CLC}_p \text{ is a conservative extension of CIC.} \]

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Parametricity is just repeating properties that were already there.
Is thus $\text{CIC}_p$ completely useless?
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NO!
Is thus CIC$_p$ completely useless?

NO!

We can still exploit the additional structure to prove independence results!
Is thus $\text{CIC}_p$ completely useless?

**No!**

We can still exploit the additional structure to prove independence results!

**Non-classicality**

We have $\not\vdash_{\text{CIC}_p} \Pi(A : \Box). A + \neg A$.

**Corollary**

We have $\not\vdash_{\text{CIC}} \Pi(A : \Box). A + \neg A$. 
It’s extremely tedious to write parametricity proofs by hand.
The Power of Computers

It’s extremely tedious to write parametricity proofs by hand.

Thankfully, we said that this model was a compiler!

**Just implement it as a Coq plugin!**

- This was done by Keller and Lasson
- Allows to systematically generate horrible proof-terms
- Computer science rules

https://github.com/coq-community/paramcoq
Syntactic models are a handy tool
Easier to comprehend (I think)
Even simple, stupid models are interesting
Implement them for Coq as mere plugins
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Be there on Friday to enter the frightening realm of effects.
Scribitur ad narrandum, non ad probandum.

Merci de votre attention.