Double-glueing and Orthogonality: Refining Models of Linear Logic through Realizability

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PiR2

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Summary

1. Usual models
2. Double-glueing
3. Tight categories
4. More structure for richer models
Introduction

- Linear logic (∼ 1986): a fruitful decomposition of logic
- A unified framework inspired from realizability
- Better understanding of constructions underlying LL models
Orthogonality

A central technique used throughout this development: orthogonality.

**Definition**

Let $R \subseteq A \times B$ be a relation. We note $a \perp b := aRb$. For any $a \subseteq A$, we define $a^\perp \subseteq B$:

$$a^\perp := \{ b \mid \forall a \in a, a \perp b \}$$

**Usual properties**

- $a \subseteq a^{\perp \perp}$
- $a \subseteq a' \Rightarrow a'^{\perp} \subseteq a^{\perp}$
- $a^{\perp \perp \perp} = a^{\perp}$
Coherent spaces are a historical model of LL designed by Girard.

**Historical definition**

A coherent space is a pair \( R = (|R|, \bowtie_R) \) where \( \bowtie_R \) is a reflexive relation on \( |R| \).

**More structure**

\[
R \otimes S := (|R| \times |S|, \ldots)
\]
\[
R \& S := (|R| \uplus |S|, \ldots)
\]
\[
!R := (\mathcal{M}_f(|R|), \ldots)
\]

\[
\ldots
\]
Folklore definition

For \( u, v \subseteq |R| \), we pose \( u \perp v \) whenever \( |u \cap v| \leq 1 \). A coherent space is a pair \( R = (|R|, C_R) \) where \( C_R \subseteq \mathcal{P}(|R|) \), called the set of **cliques** of \( R \) is s.t. \( C_R = C_R \perp \perp \).

Structure

- \( R^\perp := (|R|, C_R^\perp) \)
- \( R \otimes S := (|R| \times |S|, (C_R \cdot C_S)^\perp) \)
- \( R \& S := (|R| \uplus |S|, C_R \times C_S) \)
- \( !R := (M_f(|R|), M_f(C_R)^\perp) \)
Finiteness spaces are a more recent LL model, and in particular of differential LL.

**Finiteness spaces**

We pose $u \perp v$ whenever $u \cap v$ is finite. A finiteness space is a pair $R = (|R|, \mathcal{F}_R)$ where $\mathcal{F}_R \subseteq \mathcal{P}(|R|)$, called the set of **finitary sets** of $R$, is s.t. $\mathcal{F}_R = \mathcal{F}_R \perp \perp$

**Structure**

- $R^\perp := (|R|, \mathcal{F}_R^\perp)$
- $R \otimes S := (|R| \times |S|, (\mathcal{F}_R \cdot \mathcal{F}_S)^\perp \perp)$
- $R \& S := (|R| \uplus |S|, \mathcal{F}_R \times \mathcal{F}_S)$
- $!R := (\mathcal{M}_f(|R|), \mathcal{M}_f(\mathcal{F}_R)^\perp \perp)$
Phase semantics is another historical (but this time complete) model of LL.

**Phase semantics**

Let $\mathcal{M}$ be a commutative monoid and $\bot \subseteq \mathcal{M}$ a pole. We pose $x \perp y$ whenever $xy \in \bot$. A fact is a subset $F \subseteq \mathcal{M}$ s.t. $F = F^{\perp\perp}$.

**Structure**

- $E^{\perp} := E^{\perp}$
- $E \otimes F := (E \cdot F)^{\perp\perp}$
- $E & F := E \cap F$
- $!E := (E \cap \{1\}^{\perp\perp} \cap \mathbb{K})^{\perp\perp}$

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### Reverse-engineering

<table>
<thead>
<tr>
<th>Base structure</th>
<th>Coherence</th>
<th>Finiteness</th>
<th>Phase</th>
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<td>Facts</td>
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- **Coherence**: $|x \cap y| \leq 1$
- **Finiteness**: $|x \cap y| < \infty$
- **Phase**: $x \cdot y \in \bot$

<table>
<thead>
<tr>
<th>$R^\perp$</th>
<th>$C_R^\perp$</th>
<th>$\mathcal{F}_R^\perp$</th>
<th>$R^\perp$</th>
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<tr>
<td>1</td>
<td>${\ast}^\perp$</td>
<td>${\ast}^\perp$</td>
<td>${1}^\perp$</td>
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<tr>
<td>$R \otimes S$</td>
<td>$(C_R \cdot C_S)^\perp$</td>
<td>$(\mathcal{F}_R \cdot \mathcal{F}_S)^\perp$</td>
<td>$(R \cdot S)^\perp$</td>
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<td>$R &amp; S$</td>
<td>$C_R \times C_S$</td>
<td>$\mathcal{F}_R \times \mathcal{F}_S$</td>
<td>$R \cap S$</td>
</tr>
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Double-glueing and orthogonality  
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We can detect a common pattern in the previous examples.

- The objects are two-parts:
  - an underlying structure (a set, a monoid, ...)
  - additional information (clique, facts, finitary sets)
- A notion of orthogonality over this information
  - restriction to closed sets $A = A^\perp\perp$
- Morphisms are underlying morphisms (a relation, an element) preserving orthogonality properties

Axiomatizing this properties permits to define the double-glueing construction.
Double-glueing: general idea

Let us consider any model. With much handwaving:

- Our new formulas will be triples \((R, U, X)\) where:
  - \(R\) is an formula of the base model
  - \(U\) is an abstract set of **proofs**
  - \(X\) is an abstract set of **counter-proofs**
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- Interpretations of \((U, X) \vdash (V, Y)\) will be
  - elements from the underlying model
  - preserving proofs (by application)
  - anti-preserving counter-proofs (by co-application)
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- With enough provisos, we can lift any structure from the base model
  - Nothing added, just refining things up
In the following, we consider:

- $\mathcal{C}$ a (categorical) model of (a subsystem of) LL
- $\bot \in \mathcal{C}$ a return type
- $\bot_R \subseteq \mathcal{C}(1, R) \times \mathcal{C}(R, \bot)$ a family of orthogonalities
The practical case: slack category

We define the slack category $\mathcal{S}$ as follows:

- Objects are triples $A = (R, U, X)$ where
  - $R \in \mathbf{C}$
  - $U \subseteq \mathbf{C}(1, R)$ \iff proofs of $A$: $u \vdash^p A$
  - $X \subseteq \mathbf{C}(R, \bot)$ \iff counter-proofs of $A$: $x \vdash^o A$
  - $U \perp X$

- Morphisms $f: \mathcal{S}(A, B)$ are $f: \mathbf{C}(R, S)$ s.t.
  - $\forall u \vdash^p A, u; f \vdash^p B$ (i.e. $f(U) \subseteq V$)
  - $\forall y \vdash^o B, f; y \vdash^o A$ (i.e. $f^{-1}(Y) \subseteq X$)
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Examples of orthogonalities

- In any category, let \( \bot \subseteq C(1, \bot) \) and pose \( u \perp x \) whenever \( u; x \in \bot \)
  - These are the **focussed** orthogonalities
  - The best case for compatibility properties
- In the category \( \text{Rel} \) of sets and relations:
  - \( \text{Rel}(1, R) \cong \text{Rel}(R, \bot) \cong \mathcal{P}(R) \)
  - \( u \perp x \) whenever \( u \cap x \) at most a singleton
  - \( u \perp x \) whenever \( u \cap x \) is finite
If $C$ has some structure one can transport it onto $S$:

$$(R, U, X) \ast (S, V, Y) \equiv (R \ast S, W, Z)$$

We need to define $W$ and $Z$ accordingly!

- in particular $W \perp Z$
Lifting the structure: general case

- If $C$ has some structure one can transport it onto $S$:

  $$(R, U, X) \ast (S, V, Y) \equiv (R \ast S, W, Z)$$

- We need to define $W$ and $Z$ accordingly!
  - in particular $W \perp Z$
  - the morphisms associated to $\ast$ may be lifted to $S$ too
    - provided some well-behavedness conditions on $\perp$
    - ... and $S$ shall inherit the structure from $C$ for free!
Lifting the additives is the easy part: as in the intuitionnistic case!
Multiplicatives are hybrid disjunction/conjunction: lifting is asymmetric...

\[ \text{id}_1 \vdash^p 1 \]

\[ u_1 \vdash^p A_1 \quad u_2 \vdash^p A_2 \quad \text{then} \quad u_1 \otimes u_2 \vdash^p A_1 \otimes A_2 \]

\[ \forall u \vdash^p A, \ u; w \vdash^p B \quad \forall y \vdash^o B, \ w; y \vdash^o A \quad \text{then} \quad \hat{w} \vdash^p A \rightarrow^o B \]

\[ u^* \vdash^o A^* \quad \text{then} \quad u \vdash^p A \]

\[ \text{id}_1 \perp \chi \quad \text{then} \quad \chi \vdash^o 1 \]

\[ z \vdash^o A_1 \otimes A_2 \quad \text{then} \quad \forall u_i \vdash^p A_i, \ z[u_i] \vdash^o A_j \]

\[ u \vdash^p A \quad y \vdash^o B \quad \text{then} \quad u \cdot y \vdash^o A \rightarrow^o B \]

\[ x^* \vdash^p A^* \quad \text{then} \quad x \vdash^o A \]
Remark: To realizability fanboys

In intuitionistic realizability:

\[ f \vdash A \Rightarrow B := \forall u \vdash A, u :: f \vdash B \]

Here, a totally symmetric system:

\[ f \vdash A \to B := \begin{cases} 
    \forall u \vdash A, u :: f \vdash B \\
    \forall y \vdash B^*, f :: y \vdash A^* 
\end{cases} \]

This comes from the absence of double-orthogonal closure.
Actually we need some requirements on the orthogonality to preserve structure. (But this is ugly.)

- Whenever it is focussed, everything works
- Coherent and finiteness orthogonalities do work too
Lifting the structure: Exponentials

- We need a compatible transformation $\kappa_R : C(1, R) \to C(1, !R)$
- There is no unicity of such a transformation...
  - yet a canonical one: $\kappa(u) = 1 \xrightarrow{m} !1 \xrightarrow{!u} !R$
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  - yet a canonical one: \( \kappa(u) = 1 \xrightarrow{m} !1 \xrightarrow{!u} !R \)

\[
\frac{u \vdash^p A}{\kappa(u) \vdash^p !A}
\]

\[
\frac{x \vdash^o A}{\varepsilon; x \vdash^o !A} \quad \frac{\chi \vdash^o 1}{e; \chi \vdash^o !A} \quad \frac{z \vdash^o !A \otimes !A}{d; z \vdash^o !A}
\]

where \( \varepsilon : C(!R, R) \), \( e : C(!R, 1) \) and \( d : C(!R, !R \otimes !R) \).
In $\mathbf{Rel}$, take $!A = \mathcal{M}_{fin}(A)$
- free commutative comonoid

Canonical transformation is:

$$\kappa(u) = \{\mu \in \mathcal{M}_{fin}(A) \mid |\mu| \subseteq u\}$$

- sounds familiar:
  - similar to multiset-$\mathbf{Coh}$
  - similar to $\mathbf{Fin}$
Non-uniform exponentials

- The previous construction is defined pointwise:

  \[ \kappa(U) = \{ \kappa(u) \mid u \in U \} \]

- but \( \kappa \) can also be defined on whole sets
  - non-uniform exponentials, inspired by game semantics
  - close to explain phase semantics exponential
  - requirements less strict than the pointwise case (inclusion vs. equality)
Towards tight categories

- The slack construction is not satisfactory enough:
  - Very few examples from the literature
  - Still a lot of junk lying around
- But we did not reach our classical examples yet.
- We forgot a requirement: the closedness of (counter-)proofs sets by bi-orthogonality
- Worse is better!
Tight category

The tight category $\mathcal{T}$ is the restriction of $\mathcal{S}$ to objects of the form $(R, U \perp \perp, U \perp)$. 
The tight category $\mathcal{T}$ is the restriction of $\mathcal{S}$ to objects of the form $(R, U^\perp, U^\perp)$.

In a tight category, the set of counter-proofs is entirely defined by the set of proofs, and conversely.
A bit of polarization

Polarized objects

We define the class $\mathbb{P}$ of **positive** objects which are of the form

\[(R, U, U^\perp)\]

and dually, the class $\mathbb{N}$ of **negative** objects:

\[(R, X^\perp, X)\]

Shifts

We pose:

- $\downarrow(R, U, X) := (R, U, U^\perp) \in \mathbb{P}$
- $\uparrow(R, U, X) := (\downarrow(R, U, X)^*)^* = (R, X^\perp, X) \in \mathbb{N}$
Theorem

Positive connectives are positive (and dually), that is:

- \( \downarrow 1 = 1 \)
- \( \downarrow (A \otimes B) = \downarrow A \otimes \downarrow B \)
- \( \downarrow 0 = 0 \)
- \( \downarrow (A \oplus B) = \downarrow A \oplus \downarrow B \)

(In particular, exponentials are not polarized.)

Remark

This implies that \( \mathbb{P} \) is stable by positive connectives.
A nice drawing (or: why is linear logic depolarized)

\[ \downarrow \quad S \quad \uparrow \]

\[ (U, U^\perp) \quad (X^\perp, X) \]

LL
Tight category and lifting

To stay in the tight category, we need to dual-close everyone out:

- \(1_T := \uparrow 1_S\) and \(\bot_T := \downarrow \bot_S\)
- \(A \otimes_T B := \uparrow (A \otimes_S B)\) and \(A \bowtie_T B := \downarrow (A \bowtie_S B)\)
- \(0_T := \uparrow 0_S\) and \(\top_T := \downarrow \top_S\)
- \(A \oplus_T B := \uparrow (A \oplus_S B)\) and \(A \&_T B := \downarrow (A \&_S B)\)
- \(!_T A := \uparrow \downarrow !_S A\) and \(?_T A := \downarrow \uparrow ?_S A\)

**Theorem**

\(T\) is a model of linear logic (and this class of models is complete).
Revisiting our models

Now we can describe our three leading examples through tight categories.

- Coherent spaces is the tight category over $\text{Rel}$ with
  \[ u \perp_{\text{Coh}} x \equiv |u \cap x| \leq 1 \]

- Phase semantics on $(\mathcal{M}, \perp)$ is the tight category over the one-object category $\mathcal{C}_\mathcal{M}$ with the $\perp$-focussed orthogonality

- Finiteness spaces is the tight category over $\text{Rel}$ with
  \[ u \perp_{\text{Fin}} x \equiv |u \cap x| < \infty \]
Shifts are embedded with nice categorical properties
- $\downarrow$ is a comonad (and $\uparrow$ a monad)
- Positive objects are exactly co-algebras of $\downarrow$
- Well known adjunctions from game semantics

$$\mathbb{P}(P, \downarrow A) \cong C^+(P, A)$$
$$\mathbb{N}(\uparrow A, N) \cong C^-(A, N)$$

Unclear relationship between $\mathbb{T}_1$ and $\mathbb{T}_2$ when $\mathbb{Coh} \subseteq \mathbb{Fin}$: Hyvernats’s functor $\Phi : \mathbb{Coh} \rightarrow \mathbb{Fin}$
Subtyping

Subtypes

For any base type $R$, there is a natural order on the glued types:

$$(R, U_1, X_1) \leq (R, U_2, X_2) := U_1 \subseteq U_2 \land X_2 \subseteq X_1$$

With this order, $R$-types are a complete lattice and connectives have the expected variance.
Currently trying to integrate dependent types in Linear Logic.

- Intuition suggests that
  - $\Sigma x : A.B$ is a dependent version of $\otimes$
  - $\Pi x : A.B$ is a dependent version of $\rightarrow$
  - in particular $\Pi x : A.B := (\Sigma x . A.B^*)^*$
- In a polarized setting:

\[
\begin{align*}
  u \otimes v \vdash^p \Sigma x : A.B & := u \vdash^p A \land v \vdash^p B[u] \\
  z \vdash^o \Sigma x : A.B & := \forall u \vdash^p A, z[u] \vdash^o B[u]
\end{align*}
\]

- More natural to have a symmetrical dependence $x : A \bowtie y : B$
- A linear equality type: $(R, \{u\}, \{u\}^\perp)$
Open problems

- A handy syntax for linear logic does not exist yet
  - I do not want to work with ludics...
  - nor with proofnets!
  - $\lambda\mu\tilde{\mu}$-like systems are hard to manipulate
- I lied: phase semantics is only a degeneracy of double-glueing
  $\rightarrow$ it is proof-irrelevant, every morphism is collapsed onto 1
- What is the exact relationship between reduction/conversion and shifts?
  $\uparrow$ is a sort of lazy constructor
  conversion only at elimination?
Conclusion

- A powerful construction
  - Instanciates many interesting models
- A bit too abstract (*usine à gaz ?*)
- Not very useful in the intuitionistic case
- A tool to design new models from scratch
  - that capture interesting behaviours
Thank you for listening, folks.