Double-glueing and Orthogonality: Refining Models of Linear Logic through Realizability

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Summary

Introduction

- Linear logic ($\sim$ 1986): a fruitful decomposition of logic
- Double-glueing: Hyland and Schalk (2002)
- A unified framework inspired from realizability
- Better understanding of constructions underlying LL models

Orthogonality

A central technique used throughout this developpement: orthogonality.

Definition

Let $R \subseteq A \times B$ be a relation. We note $a \perp b := aRb$. For any $\mathfrak{a} \subseteq A$, we define $\mathfrak{a}^{\perp} \subseteq B$:

$$\mathfrak{a}^{\perp} := \{b \mid \forall a \in \mathfrak{a}, a \perp b\}$$

Usual properties

- $\mathfrak{a} \subseteq \mathfrak{a}^{\perp\perp}$
- $\mathfrak{a} \subseteq \mathfrak{a}' \Rightarrow \mathfrak{a}'^{\perp} \subseteq \mathfrak{a}^{\perp}$
- $\mathfrak{a}^{\perp\perp\perp} = \mathfrak{a}^{\perp}$

Models from the book: Coherent spaces (Historical)

Coherent spaces are a historical model of LL designed by Girard.

Historical definition

A coherent space is a pair $R = (|R|, \frown_R)$ where $\frown_R$ is a reflexive relation on $|R|$.

More structure

$$R \otimes S := (|R| \times |S|, \ldots)$$
$$R \,\&\, S := (|R| \uplus |S|, \ldots)$$
$$!R := (\mathfrak{M}_f(|R|), \ldots)$$
$$\ldots$$

Models from the book: Coherent spaces (Modern)

Folklore definition

For $u, v \subseteq |R|$, we pose $u \perp v$ whenever $|u \cap v| \leq 1$.
A coherent space is a pair $R = (|R|, \mathcal{C}_R)$ where $\mathcal{C}_R \subseteq \mathfrak{P}(|R|)$, called the set of **cliques** of R is s.t. $\mathcal{C}_R = {\mathcal{C}_R}^{\perp\perp}$.

Structure

- $R^\perp := (|R|, \mathcal{C}_R^\perp)$
- $R \otimes S := (|R| \times |S|, (\mathcal{C}_R \cdot \mathcal{C}_S)^{\perp\perp})$
- $R \,\&\, S := (|R| \uplus |S|, \mathcal{C}_R \times \mathcal{C}_S)$
- $!R := (\mathfrak{M}_f(|R|), \mathfrak{M}_f(\mathcal{C}_R)^{\perp\perp})$

Models from the book: Finiteness spaces

Finiteness spaces are a more recent LL model, and in particular of differential LL.

Finiteness spaces

We pose $u \perp v$ whenever $u \cap v$ is finite. A finiteness space is a pair $R = (|R|, \mathcal{F}_R)$ where $\mathcal{F}_R \subseteq \mathfrak{P}(|R|)$, called the set of **finitary sets** of R, is s.t. $\mathcal{F}_R = \mathcal{F}_R{}^{\perp\perp}$

Structure

- $R^{\perp} := (|R|, \mathcal{F}_R^{\perp})$
- $R \otimes S := (|R| \times |S|, (\mathcal{F}_R \cdot \mathcal{F}_S)^{\perp\perp})$
- $R \,\&\, S := (|R| \uplus |S|, \mathcal{F}_R \times \mathcal{F}_S)$
- $!R := (\mathfrak{M}_f(|R|), \mathfrak{M}_f(\mathcal{F}_R)^{\perp\perp})$

Models from the book: Phase semantics

Phase semantics is another historical (but this time complete) model of LL.

Phase semantics

Let $\mathcal{M}$ be a commutative monoid and $\perp\!\!\!\perp \subseteq \mathcal{M}$ a pole. We pose $x \perp y$ whenever $xy \in \perp\!\!\!\perp$. A **fact** is a subset $F \subseteq \mathcal{M}$ s.t. $F = F^{\perp\perp}$.

Structure

- $E^{\perp} := E^{\perp}$
- $E \otimes F := (E \cdot F)^{\perp\perp}$
- $E \,\&\, F := E \cap F$
- $!E := (E \cap \{1\}^{\perp\perp} \cap \mathbb{K})^{\perp\perp}$

Reverse-engineering

	Coherence	Finiteness	Phase
Base structure	Relations	Relations	Monoid
Topping	Cliques	Finitary sets	Facts
Orthogonality	$\lvert x \cap y \rvert \leq 1$	$\lvert x \cap y \rvert < \infty$	$x \cdot y \in \perp\!\!\!\perp$
$R^\perp$	$\mathcal{C}_R^\perp$	$\mathcal{F}_R^\perp$	$R^\perp$
1	$\{*\}^{\perp\perp}$	$\{*\}^{\perp\perp}$	$\{1\}^{\perp\perp}$
$R \otimes S$	$(\mathcal{C}_R \cdot \mathcal{C}_S)^{\perp\perp}$	$(\mathcal{F}_R \cdot \mathcal{F}_S)^{\perp\perp}$	$(R \cdot S)^{\perp\perp}$
$R \,\&\, S$	$\mathcal{C}_R \times \mathcal{C}_S$	$\mathcal{F}_R \times \mathcal{F}_S$	$R \cap S$

We can detect a common pattern in the previous examples.

- The objects are two-parts:
  - an underlying structure (a set, a monoid, ...)
  - additional information (clique, facts, finitary sets)
- A notion of orthogonality over this information
  - restriction to closed sets $A = A^{\perp\perp}$
- Morphisms are underlying morphisms (a relation, an element) preserving orthogonality properties

Axiomatizing this properties permits to define the double-glueing construction.

Double-glueing: general idea

Let us consider any model. With much handwaving:

- Our new formulas will be triples (R, U, X) where:
  - R is an formula of the base model
  - U is an abstract set of **proofs**
  - X is an abstract set of **counter-proofs**

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  - **anti-preserving** counter-proofs (by co-application)
- With enough provisos, we can lift any structure from the base model
  - Nothing added, jush refining things up

Prerequisites

In the following, we consider:

- $\mathbf{C}$ a (categorical) model of (a subsystem of) LL
- $\bot \in \mathbf{C}$ a return type
- $\bot_R \subseteq \mathbf{C}(1, R) \times \mathbf{C}(R, \bot)$ a family of orthogonalities

The practical case: slack category

We define the slack category $\mathbb{S}$ as follows:

- Objects are triples $A = (R, U, X)$ where
  - $R \in \mathbf{C}$
  - $U \subseteq \mathbf{C}(1, R)$ $\rightsquigarrow$ proofs of A: $u \Vdash^{p} A$
  - $X \subseteq \mathbf{C}(R, \bot)$ $\rightsquigarrow$ counter-proofs of A: $x \Vdash^{o} A$
  - $U \perp X$

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  - $U \perp X$
- Morphisms $f : \mathbb{S}(A, B)$ are $f : \mathbf{C}(R, S)$ s.t.
  - $\forall u \Vdash^{p} A, u; f \Vdash^{p} B$ (i.e. $f(U) \subseteq V$)
  - $\forall y \Vdash^{o} B, f; y \Vdash^{o} A$ (i.e. $f^{-1}(Y) \subseteq X$)

Examples of orthogonalities

- In any category, let $\perp\!\!\!\perp \subseteq \mathbf{C}(1, \bot)$ and pose $u \perp x$ whenever $u; x \in \perp\!\!\!\perp$
  - These are the **focussed** orthogonalities
  - The best case for compatibility properties
- In the category $\mathbf{Rel}$ of sets and relations:
  - $\mathbf{Rel}(1, R) \cong \mathbf{Rel}(R, \bot) \cong \mathfrak{P}(R)$
  - $u \perp x$ whenever $u \cap x$ at most a singleton
  - $u \perp x$ whenever $u \cap x$ is finite

Lifting the structure: general case

- If $\mathbf{C}$ has some structure one can transport it onto $\mathbb{S}$:

$$(R, U, X) * (S, V, Y) \equiv (R * S, W, Z)$$

- We need to define W and Z accordingly!
  - in particular $W \perp Z$

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- We need to define W and Z accordingly!
  - in particular $W \perp Z$
- the morphisms associated to $*$ may be lifted to $\mathbb{S}$ too
  - provided some well-behavedness conditions on $\perp$
  - ... and $\mathbb{S}$ shall inherit the structure from $\mathbf{C}$ for free!

Lifting the structure: Additives

Lifting the additives is the easy part: as in the intuitionnistic case!

$$\frac{}{\top_1 \Vdash^p \top} \qquad \frac{}{0_\bot \Vdash^o 0}$$

$$\frac{u_1 \Vdash^p A_1 \qquad u_2 \Vdash^p A_2}{\langle u_1 \mid u_2 \rangle \Vdash^p A_1 \,\&\, A_2} \qquad \frac{x_i \Vdash^o A_i}{\pi_i; x_i \Vdash^o A_1 \,\&\, A_2}$$

$$\frac{u_i \Vdash^p A_i}{u_i; \iota_i \Vdash^p A_1 \oplus A_2} \qquad \frac{x_1 \Vdash^o A_1 \qquad x_2 \Vdash^o A_2}{[x_1 \mid x_2] \Vdash^o A_1 \oplus A_2}$$

Lifting the structure: Multiplicatives

Multiplicatives are hybrid disjunction/conjunction: lifting is asymmetric...

$$\frac{}{\mathrm{id}_1 \Vdash^{p} 1} \qquad \frac{\mathrm{id}_1 \perp \chi}{\chi \Vdash^{o} 1}$$

$$\frac{u_1 \Vdash^{p} A_1 \qquad u_2 \Vdash^{p} A_2}{u_1 \otimes u_2 \Vdash^{p} A_1 \otimes A_2} \qquad \frac{\forall u_i \Vdash^{p} A_i,\ z[u_i] \Vdash^{o} A_j}{z \Vdash^{o} A_1 \otimes A_2}$$

$$\frac{\forall u \Vdash^{p} A,\ u; w \Vdash^{p} B \qquad \forall y \Vdash^{o} B,\ w; y \Vdash^{o} A}{\hat{w} \Vdash^{p} A \multimap B} \qquad \frac{u \Vdash^{p} A \qquad y \Vdash^{o} B}{u \cdot y \Vdash^{o} A \multimap B}$$

$$\frac{u^* \Vdash^{o} A^*}{u \Vdash^{p} A} \qquad \frac{x^* \Vdash^{p} A^*}{x \Vdash^{o} A}$$

Remark: To realizability fanboys

In intuitionnistic realizability:

$$f \Vdash A \Rightarrow B := \forall u \Vdash A, u :: f \Vdash B$$

Here, a totally symmetric system

$$f \Vdash A \multimap B := \left\{ \begin{array}{l} \forall u \Vdash A, u :: f \Vdash B \\ \forall y \Vdash B^*, f :: y \Vdash A^* \end{array} \right.$$

This comes from the absence of double-orthogonal closure.

Remark: Compatibility requirements

Actually we need some requirements on the orthogonality to preserve structure. (But this is ugly.)

- Whenever it is focussed, everything works
- Coherent and finiteness orthogonalities do work too

Lifting the structure: Exponentials

- We need a compatible transformation $\kappa_R : \mathbf{C}(1, R) \to \mathbf{C}(1, !R)$
- There is no unicity of such a transformation...
  - yet a canonical one: $\kappa(u) = 1 \xrightarrow{m} !1 \xrightarrow{!u} !R$

Lifting the structure: Exponentials

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  - yet a canonical one: $\kappa(u) = 1 \xrightarrow{m} !1 \xrightarrow{!u} !R$

$$\frac{u \Vdash^{p} A}{\kappa(u) \Vdash^{p} !A}$$

$$\frac{x \Vdash^{o} A}{\varepsilon; x \Vdash^{o} !A} \qquad \frac{\chi \Vdash^{o} 1}{e; \chi \Vdash^{o} !A} \qquad \frac{z \Vdash^{o} !A \otimes !A}{d; z \Vdash^{o} !A}$$

where $\varepsilon : \mathbf{C}(!R, R)$, $e : \mathbf{C}(!R, 1)$ and $d : \mathbf{C}(!R, !R \otimes !R)$.

An Enlighting Example

- In $\mathbf{Rel}$, take $!A = \mathcal{M}_{fin}(A)$
  - free commutative comonoid
- Canonical transformation is:

$$\kappa(u) = \{\mu \in \mathcal{M}_{fin}(A) \mid |\mu| \subseteq u\}$$

- sounds familiar:
  - similar to multiset-$\mathbf{Coh}$
  - similar to $\mathbf{Fin}$

Non-uniform exponentials

- The previous construction is defined pointwise:

$$\kappa(U) = \{\kappa(u) \mid u \in U\}$$

- but κ can also be defined on whole sets
  - non-uniform exponentials, inspired by game semantics
  - close to explain phase semantics exponential
  - requirements less strict than the pointwise case (inclusion vs. equality)

Towards tight categories

- The slack construction is not satisfactory enough:
  - Very few examples from the litterature
  - Still a lot of junk lying around
- But we did not reach our classical examples yet.
- We forgot a requirement: the **closedness** of (counter-)proofs sets by bi-orthogonality
- Worse is better !

Tight categories

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The tight category $\mathbb{T}$ is the **restriction** of $\mathbb{S}$ to objects of the form $(R, U^{\perp\perp}, U^{\perp})$.

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The tight category $\mathbb{T}$ is the **restriction** of $\mathbb{S}$ to objects of the form $(R, U^{\perp\perp}, U^{\perp})$.

In a tight category, the set of counter-proofs is entirely defined by the set of proofs, and conversely.

A bit of polarization

Polarized objects

We define the class $\mathbb{P}$ of **positive** objects which are of the form

$$(R, U, U^{\perp})$$

and dually, the class $\mathbb{N}$ of **negative** objects:

$$(R, X^{\perp}, X)$$

Shifts

We pose:

- $\downarrow(R, U, X) := (R, U, U^{\perp}) \in \mathbb{P}$
- $\uparrow(R, U, X) := (\downarrow(R, U, X)^*)^* = (R, X^{\perp}, X) \in \mathbb{N}$

The Meaning of Life, part XLII

Theorem

Positive connectives are positive (and dually), that is:

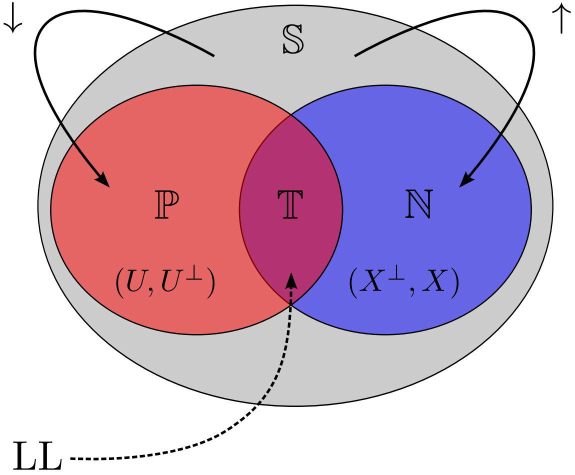
- $\downarrow 1 = 1$
- $\downarrow(A \otimes B) = \downarrow A \otimes \downarrow B$
- $\downarrow 0 = 0$
- $\downarrow(A \oplus B) = \downarrow A \oplus \downarrow B$

(In particular, exponentials are **not** *polarized.)*

Remark

This implies that $\mathbb{P}$ is stable by positive connectives.

A nice drawing (or: why is linear logic depolarized)



Tight category and lifting

To stay in the tight category, we need to dual-close everyone out:

- $1_{\mathbb{T}} := \uparrow 1_{\mathbb{S}}$ and $\bot_{\mathbb{T}} := \downarrow \bot_{\mathbb{S}}$
- $A \otimes_{\mathbb{T}} B := \uparrow(A \otimes_{\mathbb{S}} B)$ and $A \parr_{\mathbb{T}} B := \downarrow(A \parr_{\mathbb{S}} B)$
- $0_{\mathbb{T}} := \uparrow 0_{\mathbb{S}}$ and $\top_{\mathbb{T}} := \downarrow \top_{\mathbb{S}}$
- $A \oplus_{\mathbb{T}} B := \uparrow(A \oplus_{\mathbb{S}} B)$ and $A \,\&_{\mathbb{T}}\, B := \downarrow(A \,\&_{\mathbb{S}}\, B)$
- $!_{\mathbb{T}} A := \uparrow\downarrow !_{\mathbb{S}} A$ and $?_{\mathbb{T}} A := \downarrow\uparrow ?_{\mathbb{S}} A$

Theorem

$\mathbb{T}$ *is a model of linear logic (and this class of models is complete).*

Revisiting our models

Now we can describe our three leading examples through tight categories.

- Coherent spaces is the tight category over $\mathbf{Rel}$ with $u \perp_{\mathbf{Coh}} x \equiv |u \cap x| \leq 1$
- Phase semantics on $(\mathcal{M}, \perp\!\!\!\perp)$ is the tight category over the one-object category $\mathbf{C}_{\mathcal{M}}$ with the $\perp\!\!\!\perp$-focussed orthogonality
- Finiteness spaces is the tight category over $\mathbf{Rel}$ with $u \perp_{\mathbf{Fin}} x \equiv |u \cap x| < \infty$

Structure for free

- Shifts are embedded with nice categorical properties
  - $\downarrow$ is a comonad (and $\uparrow$ a monad)
  - Positive objects are exactly co-algebras of $\downarrow$
  - Well known adjunctions from game semantics

$$\mathbb{P}(P, \downarrow A) \cong \mathbf{C}^{+}(P, A)$$
$$\mathbb{N}(\uparrow A, N) \cong \mathbf{C}^{-}(A, N)$$

- Unclear relationship between $\mathbb{T}_1$ and $\mathbb{T}_2$ when $\perp^1 \neq \perp^2$
  - In $\mathbf{Rel}$ with $\perp_{\mathbf{Coh}} \subseteq \perp_{\mathbf{Fin}}$: Hyvernat's functor $\Phi : \mathbf{Coh} \to \mathbf{Fin}$

Subtyping

Subtypes

For any base type R, there is a natural order on the glued types:

$$(R, U_1, X_1) \leq (R, U_2, X_2) := U_1 \subseteq U_2 \wedge X_2 \subseteq X_1$$

With this order, R-types are a complete lattice and connectives have the expected variance.

Dependent types (WIP)

Currently trying to integrate dependent types in Linear Logic.

- Intuition suggests that
  - $\Sigma x : A.B$ is a dependent version of $\otimes$
  - $\Pi x : A.B$ is a dependent version of $\multimap$
  - in particular $\Pi x : A.B := (\Sigma x.A.B^*)^*$
  - In a polarized setting:

$$u \otimes v \Vdash^{p} \Sigma x : A.B := u \Vdash^{p} A \wedge v \Vdash^{p} B[u]$$
$$z \Vdash^{o} \Sigma x : A.B := \forall u \Vdash^{p} A, z[u] \Vdash^{o} B[u]$$

- More natural to have a symmetrical dependence $x : A \parr y : B$
- A linear equality type: $(R, \{u\}, \{u\}^{\perp})$

Open problems

- A handy syntax for linear logic does not exist yet
  - I do not want to work with ludics...
  - nor with proofnets!
  - $\bar{\lambda}\mu\tilde{\mu}$-like systems are hard to manipulate
- I lied: phase semantics is only a degeneracy of double-glueing
  - $\rightarrow$ it is proof-irrelevant, every morphism is collapsed onto 1
- What is the exact relationship between reduction/conversion and shifts?
  - $\uparrow$ is a sort of `lazy` constructor
  - conversion only at elimination?

Conclusion

- A powerful construction
  - Instanciates many interesting models
- A bit too abstract (*usine à gaz* ?)
- Not very useful in the intuitionnistic case
- A tool to design new models from scratch
  - that capture interesting behaviours

Thank you for listening, folks.