The Next 700 Syntactic Models of Type Theory

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CPP
17th January 2017
A Beginner’s Tale

Historical recollection of a younger self using Coq:

— I need to prove that $\Pi x. f x = g x$ implies $f = g$ to...
— Nay, can’t do that.
— Right, I’d also like to have $\Pi e_1 e_2 : p = q. e_1 = e_2$. How...
— Nope, not possible either.
— Fine. And what about $\Pi A B : \text{Prop.} (A \leftrightarrow B) \rightarrow A = B$?
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— Sigh.

Are you kidding me? This has to be obviously true!
If you ask why, generally you get something along the lines of:

“*That’s very simple to disprove. Let’s consider the split comprehension category where the Grothendieck fibration is the well-known blue-haired syzygetic Kardashian functor and the cartesian structure is canonically given by the algebra morphisms of hyper-loremipsum ω-potatoids. It is trivially a counter-model.*”
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(Obviously up to my brain’s isomorphisms. Any resemblance to nLab is purely coincidental.)
What You’re Usually Told

If you ask why, generally you get something along the lines of:

“That’s very simple to disprove. Let’s consider the split comprehension category where the Grothendieck fibrations is the well-known blue-hair syzygetic Kardashian functor and the cartesian structure is generically given by the algebra morphisms of hyperloremipsum ω-potatoids. It is trivially a counter-model.”

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We propose something that anybody* can understand instead.
Proofs-as-programs to the rescue

What is a model?

- Takes syntax as input.
- Interprets it into some low-level language.
- Must preserve the meaning of the source.
- Refines the behaviour of under-specified structures.

Luckily we're computer scientists in here. « Oh yes, we call that a compiler... »
(Thanks, Curry-Howard!)
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But I understand well type-theory!
And I know how to write program translations.

Let’s write models as compilers from type theory into itself!
Define $\cdot$ on the syntax and derive the type interpretation $\cdot$ from it s.t.

$$\vdash M : A \quad \text{implies} \quad \vdash [M] : [A]$$
Define $[\cdot]$ on the syntax and derive the type interpretation $[\cdot]$ from it s.t.

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Obviously, that’s subtle.

- The correctness of $[\cdot]$ lies in the meta (Darn, Gödel!)
- The translation must preserve typing (Not easy)
- In particular, it must preserve conversion (Argh!)
Define $\cdot$ on the syntax and derive the type interpretation $\lbrack \cdot \rbrack$ from it s.t.

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Yet, a lot of nice consequences.

- Does not require non-type-theoretical foundations (monism)
- Can be implemented in your favourite proof assistant
- Easy to show (relative) consistency, look at $\lbrack \text{False} \rbrack$
- Easier to understand computationally
In The Remainder of This Talk

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The 578th Will Shock You!

(Just kidding. I don’t want doctors to hate me.)
Where the Wild Things Are

— What is fully specified in type theory?

- **Inductive types**, because of dependent elimination.
Where the Wild Things Are

— What is fully specified in type theory?

  • **Inductive types**, because of dependent elimination.

— What is *not* fully specified in type theory?
Everything else!

  • **Functions**: only specified w.r.t. $\beta$-reduction
  • **Co-inductive types**: only specified w.r.t. projections
  • **Universes**: only specified w.r.t. rhs of a colon
  • ...

Let’s joyfully refine the intensional behaviour of random stuff in there.
First target: functions. The only thing you know about them:

$$(\lambda x : A. M) \ N \equiv M\{x := N\}$$
Negating Functional Extensionality

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$$(\lambda x : A. M) \ N \equiv M[x := N]$$

Let’s take advantage of this by mangling functions.

$$[x] := x$$
$$[\lambda x : A. M] := (\lambda x : [A]. [M], \text{true})$$
$$[M \ N] := [M]. \pi_1 \ [N]$$
$$[\square] := \square$$
$$[\Pi x : A. B] := (\Pi x : [A]. [B]) \times \text{bool}$$
$$[\ldots] := \ldots$$
$$[A] := [A]$$
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\begin{align*}
[x] &= x \\
[\lambda x : A. M] &= (\lambda x : [A]. [M], \text{true}) \\
[M N] &= [M]. \pi_1 [N] \\
[\Box] &= \Box \\
[\Pi x : A. B] &= (\Pi x : [A]. [B]) \times \text{bool} \\
[\ldots] &= \ldots \\
[A] &= [A]
\end{align*}
\]

Obviously \( \Gamma \vdash M : A \) implies \( \Gamma \Downarrow [M] : [A] \).
Now, we interpret everything through the $[\cdot]$ translation.

- We call the source theory all terms that have some type $[A]$
- Given $M : [A]$ we can extend the source with a constant $M^\bullet : A$

\[ [M^\bullet] := M \]

- Conversion is extended the same way:

\[ M \equiv_{\text{source}} N := [M] \equiv_{\text{target}} [N] \]
Syntactically, this means that you can extend the source theory with

\[
\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda' x : A. M : \Pi x : A. B}
\]

defined as:

\[
[\lambda' x : A. M] := (\lambda x : \llbracket A \rrbracket. [M], \text{false})
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Rembember:

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\begin{align*}
[\lambda x : A. M] & \ := \ (\lambda x : \llbracket A \rrbracket. [M], \text{true}) \\
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\end{align*}
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Negating Functional Extensionality II

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Rembember:

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[\lambda x : A. M] := (\lambda x : [A]. [M], \text{true}) \\
[M N] := [M].\pi_1 [N]
\]

Clearly this new abstraction has the same behaviour as the original one.

\[
[ (\lambda' x : A. M) N ] \equiv [ M \{ x := N \} ]
\]
Now, it is easy to see how to negate functional extensionality. Consider:

$$\Sigma(fg : 1 \rightarrow 1). (\Pi i : 1. f i = g i) \land f \neq g$$
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\[ \Sigma(fg : 1 \to 1). (\Pi i : 1. f\ i = g\ i) \land f \neq g \]

This is translated into something that is essentially:

\[ \Sigma(fg : (1 \to 1) \times \text{bool}). (\Pi i : 1. f.\pi_1\ i = g.\pi_1\ i) \land f \neq g \]

(The actual translation is a little noisier, but this does not change the idea.)
Negating Functional Extensionality III

Now, it is easy to see how to negate functional extensionality. Consider:

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(The actual translation is a little noisier, but this does not change the idea.)

Take $f := [\lambda x : 1. x]$ and $g := [\lambda' x : 1. x]$, and voilá!
Where We Cheated

We did not explicit the rules of the source theory.
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In particular, it is clear that the model invalidates $\eta$-rules.

\[ [\lambda x : A. M \ x] \neq [M] \]

\[ (\lambda x : [A]. [M]. \pi_1 x, \text{true}) \neq [M] \]

It’s much harder to negate extensionality while preserving $\eta$.

(Dialectica does that.)
Stream extensionality

We can use a very similar trick to intentionalize streams. Idea:

\[
\text{stream } A := (\text{stream } [A]) \times \text{bool}
\]

This interprets all negative co-inductive properties ("co-pattern style").

And there is no reasonable \(\eta\)-rule on cofixpoints anyway.
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And there is no reasonable \( \eta \)-rule on cofixpoints anyway.

Then just as easily we show that:

\[
\sum (f g : \text{stream } 1). (\text{bisimilar } 1 f g) \land f \neq g
\]
Type Extensionality

Once again, the same trick can be applied to types.

\[
\begin{align*}
[x] & := x \\
[\lambda x : A. M] & := \lambda x : [A]. [M] \\
[M N] & := [M][N] \\
[\square_i] & := (\square_i \times \text{bool}, \text{true}) \\
[\Pi x : A. B] & := ((\Pi x : [A]. [B]), \text{true}) \\
[A] & := [A].\pi_1
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[\Pi x : A. B] & := ((\Pi x : [A].[B]), \text{true}) \\
[A] & := [A].\pi_1
\end{align*}
\]

“New types are a pair of a type and a boolean!” Tricky fixpoint:

\[
[\square_i] : [\square_{i+1}] \iff (\square_i \times \text{bool}, \text{true}) : \square_{i+1} \times \text{bool}
\]
Negating Propositional Extensionality

You can translate an impredicative universe alike:

\[ [*] := (\star \times \text{bool}, \text{true}) \]

It is still an impredicative universe!
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You can translate an impredicative universe alike:

\[[*] := (* \times \text{bool}, \text{true})\]

It is still an impredicative universe!

It is then easy to show:

\[\Sigma(P Q : *). (P \leftrightarrow Q) \land P \neq Q\]

\[
\sim \Sigma(P Q : * \times \text{bool}). (P.\pi_1 \leftrightarrow Q.\pi_1) \land P \neq Q
\]

Take for instance True and its evil twin True\(^\dagger\):

\[[\text{True}] := (\text{True}, \text{true})\]
\[[\text{True}^\dagger] := (\text{True}, \text{false})\]
Where Will They Stop?

- This shows that universes are “amorphous” in type theory.
- The only thing that matters is $\cdot$ in the translation!
- We simply used a projection here
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Let’s do **way much better** (or worse, depends on your beliefs).
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Let’s do **way much better** (or worse, depends on your beliefs).

Let’s turn Coq into Python!
The Basilisk

Idea: if $A : \Box$ then $[A] : \text{TYPE}$, the type of inductive-recursive codes!

\[
\begin{align*}
\text{Inductive TYPE :=} \\
\mid U : \text{TYPE} \\
\mid \Pi (A : \text{TYPE}), (\text{Elt } A \to \text{TYPE}) \to \text{TYPE} \\
\mid \ldots \\
\text{with Elt } (A : \text{TYPE}) := \text{match } A \text{ with} \\
\mid U \Rightarrow \text{TYPE} \\
\mid \Pi A B \Rightarrow \Pi (x : \text{Elt } A), \text{Elt } (B x) \\
\mid \ldots \\
\text{end.}
\end{align*}
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(Note: We need to stratify a bit to make this work.)
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\begin{align*}
\square & := U \\
[\Pi x : A. \,B] & := \Pi \,[A] \,(\lambda x : [A]. [B]) \\
[A] & := \text{Elt } [A]
\end{align*}
Behold!

This allows definitions by case-analysis on types!

For instance, it is now possible to define:

- \( f : \prod A : \Box. A \to A \) \quad (\sim \prod A : \text{TYPE}. \text{Elt } A \to \text{Elt } A)
- \( f \text{bool} : \text{bool} \to \text{bool} \) is negation
- \( f \ A \) is identity otherwise

Morally it is the most anti-parametric thing one can do. Abstractly:

Type theory is compatible with ad-hoc polymorphism.

(Yes, this surprised me as well.)
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For instance, it is now possible to define:

- $f : \Pi A : \Box. A \to A$ ($\sim \Pi A : \text{TYPE. Elt} A \to \text{Elt} A$)
- $f \text{bool} : \text{bool} \to \text{bool}$ is negation
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What else

We have a soundness proof in Coq for most of the previous translations.

- Based on Siles’s definition of De Bruijn implementation of CC
- “Deep embedding“
- Shows that the model preserve consistency in a easy way

There is also an experimental plugin to translate terms automagically.

https://github.com/CoqHott/Program-translations-CC-omega
Conclusion

- We’ve described a simple class of models
- Rooted in computer science POV
- Sufficient to negate a lot of extensionality principles
  - Functions
  - Co-inductive types
  - Universes
- Implemented them!
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- We advocate for this kind of models
- A few more instances from the literature
- Stay tuned!
Thanks for your attention.