The Next 700 Syntactic Models of Type Theory

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CPP
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A Beginner's Tale

Historical recollection of a younger self using Coq:

— I need to prove that $\Pi x.\, f\ x = g\ x$ implies $f = g$ to...

— Nay, can't do that.

— Right, I'd also like to have $\Pi e_1\ e_2 : p = q.\ e_1 = e_2$. How...

— Nope, not possible either.

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Are you kidding me? This *has* to be obviously true!

What You're Usually Told

If you ask why, generally you get something along the lines of:

> *"That's very simple to disprove. Let's consider the split comprehension category where the Grothendieck fibration is the well-known* ***blue-haired syzygetic Kardashian functor*** *and the cartesian structure is canonically given by the algebra morphisms of* ***hyper-loremipsum*** ω***-potatoids****. It is trivially a counter-model."*

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We propose something that anybody* can understand instead.

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« Oh yes, we call that a *compiler*... »

(Thanks, Curry-Howard!)

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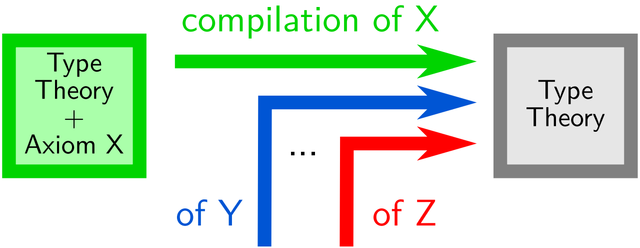
Let's write models as compilers from type theory into itself!



;

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Define $[\cdot]$ on the syntax and derive the type interpretation $[\![\cdot]\!]$ from it s.t.

$$\vdash M : A \qquad \text{implies} \qquad \vdash [M] : [\![A]\!]$$

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- The correctness of $[\cdot]$ lies in the meta (Darn, Gödel!)
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Yet, a lot of nice consequences.

- Does not require non-type-theoretical foundations (*monism*)
- Can be implemented in your favourite proof assistant
- Easy to show (relative) consistency, look at $[\![\texttt{False}]\!]$
- Easier to understand computationally

In The Remainder of This Talk

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(Just kidding. I don't want doctors to hate me.)

Where the Wild Things Are

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- **Inductive types**, because of dependent elimination.

— What is *not* fully specified in type theory?
Everything else!

- **Functions**: only specified w.r.t. β-reduction
- **Co-inductive types**: only specified w.r.t. projections
- **Universes**: only specified w.r.t. rhs of a colon
- ...

Let's joyfully refine the intensional behaviour of random stuff in there.

Negating Functional Extensionality

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Let's take advantage of this by mangling functions.

$$\begin{array}{lcl}
[x] & := & x \\
[\lambda x : A.\, M] & := & (\lambda x : [\![A]\!].\, [M], \texttt{true}) \\
[M\, N] & := & [M].\pi_1\, [N] \\
[\square] & := & \square \\
[\Pi x : A.\, B] & := & (\Pi x : [\![A]\!].\, [\![B]\!]) \times \texttt{bool} \\
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Obviously $\Gamma \vdash M : A$ implies $[\![\Gamma]\!] \vdash [M] : [\![A]\!]$.

Through The Looking Glass

Now, we interpret everything through the $[\cdot]$ translation.

- We call the source theory all terms that have some type $[\![A]\!]$
- Given $M : [\![A]\!]$ we can extend the source with a constant $M^\bullet : A$

$$[M^\bullet] := M$$

- Conversion is extended the same way:

$$M \equiv_{\texttt{source}} N := [M] \equiv_{\texttt{target}} [N]$$

Negating Functional Extensionality II

Syntactically, this means that you can extend the source theory with

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda' x : A.\, M : \Pi x : A.\, B}$$

defined as:

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Rembember:

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Clearly this new abstraction has the same behaviour as the original one.

$$[(\lambda' x : A.\, M)\, N] \equiv [M\{x := N\}]$$

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This is translated into something that is essentially:

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Take $f := [\lambda x : 1.\, x]$ and $g := [\lambda' x : 1.\, x]$, and *voilá*!

Where We Cheated

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In particular, it is clear that the model invalidates η-rules.

$$\begin{array}{ccc} [\lambda x : A.\, M\ x] & \not\equiv & [M] \\ \equiv & & \equiv \\ (\lambda x : [\![A]\!].\, [M].\pi_1\ x, \mathtt{true}) & \not\equiv & [M] \end{array}$$

It's much harder to negate extensionality while preserving η.
(Dialectica does that.)

Stream extensionality

We can use a very similar trick to intentionalize steams. Idea:

$$[\![\texttt{stream}\ A]\!] := (\texttt{stream}\ [\![A]\!]) \times \texttt{bool}$$

This interprets all negative co-inductive properties ("co-pattern style").

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Then just as easily we show that:

$$\Sigma(f\,g : \texttt{stream}\ 1).\,(\texttt{bisimilar}\ 1\ f\ g) \wedge f \neq g$$

Type Extensionality

Once again, the same trick can be applied to types.

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$$

"New types are a pair of a type and a boolean!" Tricky fixpoint:

$$[\Box_i] : [\![\Box_{i+1}]\!] \quad \Leftrightarrow \quad (\Box_i \times \texttt{bool}, \texttt{true}) : \Box_{i+1} \times \texttt{bool}$$

Negating Propositional Extensionality

You can translate an impredicative universe alike:

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It is then easy to show:

$$\begin{array}{rl} & [\![\Sigma(P\,Q : *).\,(P \leftrightarrow Q) \wedge P \neq Q]\!] \\ \sim & \Sigma(P\,Q : * \times \texttt{bool}).\,(P.\pi_1 \leftrightarrow Q.\pi_1) \wedge P \neq Q \end{array}$$

Take for instance True and its evil twin $\texttt{True}^\dagger$:

$$\begin{array}{rcl} [\texttt{True}] & := & (\texttt{True}, \texttt{true}) \\ [\texttt{True}^\dagger] & := & (\texttt{True}, \texttt{false}) \end{array}$$

Where Will They Stop?

- This shows that universes are “amorphous” in type theory
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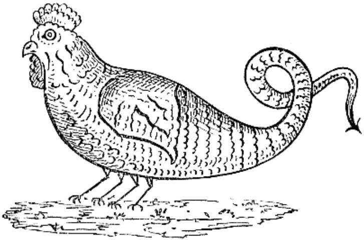
Let’s turn Coq into Python!



The Basilisk

Idea: if $A : \Box$ then $[A]$: TYPE, the type of inductive-recursive **codes**!

```
Inductive TYPE :=
| 𝒰 : TYPE
| Pi : Π (A : TYPE), (Elt A → TYPE) → TYPE
| ...
with Elt (A : TYPE) := match A with
| 𝒰 ⇒ TYPE
| Pi A B ⇒ Π (x : Elt A), Elt (B x)
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$$\begin{array}{lcl} [\square] & := & \mathcal{U} \\ [\Pi x : A.\, B] & := & \texttt{Pi}\ [A]\ (\lambda x : [\![A]\!].\, [B]) \\ [\![A]\!] & := & \texttt{Elt}\ [A] \end{array}$$

Behold!

This allows definitions by case-analysis on types!

For instance, it is now possible to define:

- $f : \Pi A : \square.\, A \to A \qquad (\sim \Pi A : \mathtt{TYPE}.\, \mathtt{Elt}\ A \to \mathtt{Elt}\ A)$
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Morally it is the most anti-parametric thing one can do. Abstractly:

Type theory is compatible with ad-hoc polymorphism.

(Yes, this surprised me as well.)

What else

We have a soundness proof in Coq for most of the previous translations.

- Based on Siles's definition of De Bruijn implementaton of CC
- "Deep embedding"
- Shows that the model preserve consistency in a easy way

There is also an experimental plugin to translate terms automagically.

`https://github.com/CoqHott/Program-translations-CC-omega`

Conclusion

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  - Functions
  - Co-inductive types
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- We advocate for this kind of models
- A few more instances from the literature
- Stay tuned!

Thanks for your attention.