Can Dialectica break bricks?

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PPS/πr^2

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Once upon a time...

- Cataclysm: Gödel's incompleteness theorem (1931)

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- Cataclysm: Gödel's incompleteness theorem (1931)

We do not fight alienation with an alienated logic.

- Justifying arithmetic differently
- ... Intuitionistic logic!
    - Double-negation translation (1933)
    - **Dialectica** (30's, published in 1958)

Plan

1. Historical presentation
2. A step into modernity
3. Enters Linear Logic
4. A syntactic presentation
5. Towards CC^ω

What is Dialectica?

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- A translation from HA into HA^ω
- That preserves intuitionistic content

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- A translation from HA into HA^ω
- That preserves intuitionistic content
- But offers two semi-classical principles:

$$\mathsf{MP}\ \frac{\neg(\forall n \in \mathbb{N}.\, \neg P\, n)}{\exists n \in \mathbb{N}.\, P\, n} \qquad \frac{(\forall n \in \mathbb{N}.\, P\, n) \to \exists m \in \mathbb{N}.\, Q\, m}{\exists m \in \mathbb{N}.\, (\forall n \in \mathbb{N}.\, P\, n) \to Q\, m}\ \mathsf{IP}$$

Parental advisory required

For the sake of exhaustivity, we'll take a glimpse at the historical presentation of Dialectica.

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Warning! Dusty logic inside

- Translation acting on formulæ
- Prevalence of negative connectives
- First-order logic
- Lots of arithmetic encoding
- Does not preserve β-reduction

Dusty logics

Dialectica, Dawn of Curry-Howard:

$$\vdash A \qquad \mapsto \qquad \vdash A^D \equiv \exists \vec{u}. \forall \vec{x}. A_D[\vec{u}, \vec{x}]$$

$$\begin{array}{c|lll}
A \wedge B & \exists \vec{u}\, \vec{v}. & \forall \vec{x}\, \vec{y}. & A_D[\vec{u}, \vec{x}] \wedge B_D[\vec{v}, \vec{y}] \\
A \vee B & \exists \vec{u}\, \vec{v}\, b. & \forall \vec{x}\, \vec{y}. & (b = 0 \wedge A_D[\vec{u}, \vec{x}]) \vee (b = 1 \wedge B_D[\vec{v}, \vec{y}]) \\
A \rightarrow B & \exists \vec{\varphi}\, \vec{\psi}. & \forall \vec{u}\, \vec{y}. & A_D[\vec{u}, \vec{\psi}(\vec{u}, \vec{y})] \rightarrow B_D[\vec{\varphi}(\vec{u}), \vec{y}] \\
 & & & \\
\forall n. A[n] & \exists \vec{\varphi}. & \forall \vec{x}\, n. & A_D[\vec{\varphi}(n), \vec{x}, n] \\
\exists n. A[n] & \exists \vec{u}\, n. & \forall \vec{x}. & A_D[\vec{u}, n, \vec{x}]
\end{array}$$

Sound translation, blah blah blah.

A step into modernity

Let us forget the 50's, and rather jump directly to the 90's.

- Take seriously the **computational** content
- Dialectica as a **typed** object
- Works of De Paiva, Hyland, etc.

Gödel's anatomy

$$\exists \vec{u}.\, \forall \vec{x}.\, A_D[\vec{u}, \vec{x}]$$

$\mathbb{W}(A)$

`Type`

Witness

$\mathbb{C}(A)$

`Type`

Counter

$\perp_A$

$\mathbb{W}(A) \to \mathbb{C}(A) \to$ `Prop`

Orthogonality

The same, with types

A proof $\vdash u : A$ is a term $\vdash u : \mathbb{W}(A)$ such that:

$$\forall x : \mathbb{C}(A).\, u \perp_A x$$

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If we wish to put more types in there:

	$\mathbb{W}$	$\mathbb{C}$
$A \wedge B$	$\exists \vec{u}\,\vec{v}.$	$\forall \vec{x}\,\vec{y}.$
$A \times B$	$\mathbb{W}(A) \times \mathbb{W}(B)$	$\mathbb{C}(A) \times \mathbb{C}(B)$
$A \vee B$	$\exists b\,\vec{u}\,\vec{v}.$	$\forall \vec{x}\,\vec{y}.$
$A + B$	$\texttt{bool} \times \mathbb{W}(A) \times \mathbb{W}(B)$	$\mathbb{C}(A) \times \mathbb{C}(B)$
$A \rightarrow B$	$\exists \vec{\varphi}\,\vec{\psi}.$	$\forall \vec{u}\,\vec{y}.$
$A \rightarrow B$	$\left\{\begin{array}{l}\mathbb{W}(A) \rightarrow \mathbb{W}(B) \\ \mathbb{C}(B) \rightarrow \mathbb{W}(A) \rightarrow \mathbb{C}(A)\end{array}\right.$	$\mathbb{W}(A) \times \mathbb{C}(B)$

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$A \to B$	$\exists \vec{\varphi}\, \vec{\psi}.$	$\forall \vec{u}\, \vec{y}.$
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A *Déjà vu*

But, grandmother, how familiar you look...

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- Classical realizability: $\mathbb{W}(A)$ proofs $|A|$, $\mathbb{C}(A)$ stacks $||A||$
- Double-orthogonality based models
- *Double-glueing*
- Reducibility candidates
- ...

Not too hastily

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Not too hastily

- We could give a computational content right now
- But it would be *ad-hoc*, inheriting from the encodings of Dialectica
- Let us use our our favorite tool: **Linear Logic**!
  - A genuine exponential!
  - With real chunks of sum types!

Linearized Dialectica

As forecasted on the previous slide, we essentially apply the following modifications:

- Introduction of duality with sum types
- Call-by-name decomposition of the arrow:

$$A \to B \quad \equiv \quad !A \multimap B$$

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- Introduction of duality with sum types
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$\boxed{\text{Now we will be translating } LL \text{ formulæ into } LJ \text{ ones.}}$

Requirements

We will be interpreting the formulæ of linear logic:

$$A, B ::= A \otimes B \mid A \parr B \mid A \oplus B \mid A \,\&\, B \mid\, !A \mid\, ?A$$

It is therefore sufficient to define $\color{blue}{\mathbb{W}}(A)$, $\color{red}{\mathbb{C}}(A)$ and $\perp_A$ for each A, where:

$$\perp_A \subseteq \color{blue}{\mathbb{W}}(A) \times \color{red}{\mathbb{C}}(A)$$

Forget the dual

Taking inspiration from the double-orthogonality models, we require:

- $\mathbb{W}(A^{\perp}) \equiv \mathbb{C}(A)$ and conversely;

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Taking inspiration from the double-orthogonality models, we require:

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$\leadsto$ It is sufficient to define our structures on positive types

$\leadsto$ We will get them for dual connectives... by duality.

We define therefore:

$$\frac{u \not\perp_A x}{x \perp_{A^{\perp}} u}$$

Sum types

	$\mathbb{W}$	$\mathbb{C}$
$A \times B$	$\mathbb{W}(A) \times \mathbb{W}(B)$	$\mathbb{C}(A) \times \mathbb{C}(B)$
$A \mathbin{\&} B$	$\mathbb{W}(A) \times \mathbb{W}(B)$	$\mathbb{C}(A) + \mathbb{C}(B)$
$A + B$	$\mathtt{bool} \times \mathbb{W}(A) \times \mathbb{W}(B)$	$\mathbb{C}(A) \times \mathbb{C}(B)$
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$A \oplus B$	$\mathbb{W}(A) + \mathbb{W}(B)$	$\mathbb{C}(A) \times \mathbb{C}(B)$

$$\frac{v \perp_A z_2}{\texttt{inr}\, v \perp_{A \oplus B} (z_1, z_2)} \qquad \frac{u \perp_A z_1}{\texttt{inl}\, u \perp_{A \oplus B} (z_1, z_2)}$$

Linear decomposition

	$\mathbb{W}$	$\mathbb{C}$
$A \to B$	$\begin{cases} \mathbb{W}(A) \to \mathbb{W}(B) \\ \mathbb{C}(B) \to \mathbb{W}(A) \to \mathbb{C}(A) \end{cases}$	$\mathbb{C}(A) \times \mathbb{C}(B)$
$A \multimap B$	$\begin{cases} \mathbb{W}(A) \to \mathbb{W}(B) \\ \mathbb{C}(B) \to \mathbb{C}(A) \end{cases}$	$\mathbb{W}(A) \times \mathbb{C}(B)$
$!A$	$\mathbb{W}(A)$	$\mathbb{W}(A) \to \mathbb{C}(A)$

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$$\frac{u \perp_A \psi\, y \quad \to \quad \varphi\, u \perp_B y}{(\varphi, \psi) \perp_{A \multimap B} (u, y)} \qquad \frac{u \perp_A z\, u}{u \perp_{!A} z}$$

Handwaving justification

- The interpretation of arrow forces its reversibility:

$$A \multimap B \cong B^\perp \multimap A^\perp$$

  $\rightsquigarrow$ Like the two-way proofnet wires

Handwaving justification

- The interpretation of arrow forces its reversibility:

$$A \multimap B \cong B^{\perp} \multimap A^{\perp}$$

  $\leadsto$ Like the two-way proofnet wires
- The bang connective is a *shift* :
  $\leadsto$ Opponent may wait for the player to play and inspect its answer
- Duality is rôle swapping

About linearity

We're not linear by chance.

[1] Assuming we've defined 1.

[2] May contain nuts.

We're not linear by chance.

Indeed, in Dialectica, we do not have the following morphisms:

$$\vdash A \multimap 1^{1}$$

$$\vdash A \multimap A \otimes A$$

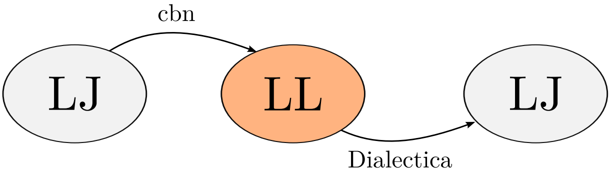
Hence we have true linear constraints![2]

[1] Assuming we've defined 1.

[2] May contain nuts.

Intepretation of the call-by-name λ-calculus

We're now trying to translate the λ-calculus through Dialectica.



- First through the call-by-name linear decomposition into LL;
- Then into LJ with the linear Dialectica.

Brief reminder

We recall here the call-by-name translation of the λ-calculus into LL:

$$[\![A \to B]\!] \equiv ![\![A]\!] \multimap [\![B]\!]$$

$$[\![A \times B]\!] \equiv ![\![A]\!] \otimes ![\![B]\!]$$

$$[\![A + B]\!] \equiv ![\![A]\!] \oplus ![\![B]\!]$$

$$[\![\Gamma \vdash A]\!] \equiv \bigotimes ![\![\Gamma]\!] \vdash [\![A]\!]$$

In order to interpret the λ-calculus, we need the following:

Dummy term

For all type A, there exists $\vdash \varnothing_A : \mathbb{W}(A)$.

Decidability of the orthogonality

The $\perp_A$ relation is decidable. In particular, there must exist some λ-term

$$@^A : \mathbb{W}(A) \to \mathbb{W}(A) \to \mathbb{C}(A) \to \mathbb{W}(A)$$

with the following behaviour:

$$u_1 @^A_x u_2 \cong \texttt{if } u_1 \perp_A x \texttt{ then } u_2 \texttt{ else } u_1$$

Did you solve the organization issue?

If we were to use the translation as is, we would bump up into an unbearable bureaucracy. Instead, we are going to use the following isomorphism.

$$[\![x_1 : \Gamma_1, \ldots x_n : \Gamma_n \vdash t : A]\!] \cong \mathbb{W}(\Gamma) \to \begin{cases} \mathbb{W}(A) \\ \mathbb{C}(A) \to \mathbb{C}(\Gamma_1) \\ \vdots \\ \mathbb{C}(A) \to \mathbb{C}(\Gamma_n) \end{cases}$$

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Which results in the following translations:

$$[\![\vec{x} : \Gamma \vdash t : A]\!] \equiv \begin{cases} \vec{x} : \mathbb{W}(\Gamma) \vdash t^\bullet : \mathbb{W}(A) \\ \vec{x} : \mathbb{W}(\Gamma) \vdash t_{x_1} : \mathbb{C}(A) \to \mathbb{C}(\Gamma_1) \\ \vdots \\ \vec{x} : \mathbb{W}(\Gamma) \vdash t_{x_n} : \mathbb{C}(A) \to \mathbb{C}(\Gamma_n) \end{cases}$$

Translation

For $(-)^{\bullet}$:

$$\begin{array}{lcl} x^{\bullet} & \equiv & x \\ (\lambda x.\, t)^{\bullet} & \equiv & \left\{ \begin{array}{l} \lambda x.\, t^{\bullet} \\ \lambda \pi x.\, t_x\, \pi \end{array} \right. \\ (t\, u)^{\bullet} & \equiv & (\texttt{fst}\ t^{\bullet})\, u^{\bullet} \end{array}$$

Translation II

For t_x :

$$\begin{array}{lcl}
x_x & \equiv & \lambda\pi.\,\pi \\
 & : & \mathbb{C}(A) \rightarrow \mathbb{C}(A) \\
y_x & \equiv & \lambda\pi.\,\varnothing \\
 & : & \mathbb{C}(A) \rightarrow \mathbb{C}(\Gamma_i) \\
(\lambda y.\,t)_x & \equiv & \lambda(y, \pi).\,t_x\,\pi \\
 & : & \mathbb{W}(A) \times \mathbb{C}(B) \rightarrow \mathbb{C}(\Gamma_i) \\
(t\,u)_x & \equiv & \lambda\pi.\,u_x\,((\texttt{snd}\ t^\bullet)\,\pi\,u^\bullet)\ @_\pi\ t_x\,(u^\bullet, \pi) \\
 & : & \mathbb{C}(B) \rightarrow \mathbb{C}(\Gamma_i)
\end{array}$$

It just works... Does it?

Soundness

If $\vdash t : A$, then $\vdash [\![t]\!] : \mathbb{W}(A)$, and in addition, for all $\pi : \mathbb{C}(A)$, $t \perp_A \pi$.

It just works... Does it?

Soundness

If $\vdash t : A$, then $\vdash [\![t]\!] : \mathbb{W}(A)$, and in addition, for all $\pi : \mathbb{C}(A)$, $t \perp_A \pi$.

Sadness

The translation is still not stable by β-reduction.

Almost there

$\boxed{\text{Using } \varnothing \text{ and } @ \text{ is another encoding of Dialectica.}}$

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- We want **lists!** almost…
- We just change:

$$\begin{array}{rcl} \mathbb{C}(!A) & \equiv & \mathbb{W}(A) \to \mathbb{C}(A) \\ \mathbb{C}(!A) & \equiv & \mathbb{W}(A) \to \texttt{list}\ \mathbb{C}(A) \end{array}$$

- Term interpretation is almost unchanged:
  - $\varnothing$ becomes the empty list;
  - $@$ becomes concatenation
  - … plus a bit of monadic boilerplate
- We do not need orthogonality anymore…

What about the computational content?

This gives us the following types for the translation:

$$[\![\vec{x} : \Gamma \vdash t : A]\!] \equiv \begin{cases} \vec{x} : \mathbb{W}(\Gamma) \vdash t^{\bullet} : \mathbb{W}(A) \\ \vec{x} : \mathbb{W}(\Gamma) \vdash t_{x_1} : \mathbb{C}(A) \to \texttt{list}\ \mathbb{C}(\Gamma_1) \\ \vdots \\ \vec{x} : \mathbb{W}(\Gamma) \vdash t_{x_n} : \mathbb{C}(A) \to \texttt{list}\ \mathbb{C}(\Gamma_n) \end{cases}$$

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- $t^\bullet$ is clearly the lifting of t;
- What on earth is t_{x_i}?

An unbearable suspense

A small interlude of ~~advertisement~~ **definitions** to introduce you to the KAM.

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Closures	c	$::=$	(t, σ)
Environments	σ	$::=$	$\emptyset \mid \sigma + (x := c)$
Stacks	π	$::=$	$\varepsilon \mid c \cdot \pi$
Processes	p	$::=$	$\langle c \mid \pi \rangle$

Push	$\langle (t\, u, \sigma) \mid \pi \rangle$	$\rightarrow$	$\langle (t, \sigma) \mid (u, \sigma) \cdot \pi \rangle$
Pop	$\langle (\lambda x.\, t, \sigma) \mid c \cdot \pi \rangle$	$\rightarrow$	$\langle (t, \sigma + (x := c)) \mid \pi \rangle$
Grab	$\langle (x, \sigma + (x := c)) \mid \pi \rangle$	$\rightarrow$	$\langle c \mid \pi \rangle$
Garbage	$\langle (x, \sigma + (y := c)) \mid \pi \rangle$	$\rightarrow$	$\langle (x, \sigma) \mid \pi \rangle$

The Krivine Machine™

Closures all the way down

Let:

- a term $\vec{x} : \Gamma \vdash t : A$
- a closure $\sigma \vdash \Gamma$
- a stack $\vdash \pi : A^{\perp}$ (i.e. $[\![\pi]\!] : \mathbb{C}(A)$)

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- a term $\vec{x} : \Gamma \vdash t : A$
- a closure $\sigma \vdash \Gamma$
- a stack $\vdash \pi : A^{\perp}$ (i.e. $[\![\pi]\!] : \mathbb{C}(A)$)

Then $t_{x_i}\,\pi$ is the list made of **the stacks encountered by** x_i while evaluating $\langle (t, \sigma) \mid \pi \rangle$, i.e.

$$(t_{x_i}\{\vec{x} := \sigma\})\,\pi = [\rho_1; \ldots; \rho_m]$$

$$\begin{array}{ccc}
\langle (t, \sigma) \mid \pi \rangle & \longrightarrow^* & \langle (x_i, \sigma_1) \mid \rho_1 \rangle \\
 & \vdots & \vdots \\
 & \longrightarrow^* & \langle (x_i, \sigma_m) \mid \rho_m \rangle
\end{array}$$

Otherwise said, Dialectica tracks the Grab rule.

Look

$$
\begin{array}{lcl}
x_x & \equiv & \lambda\pi.\,[\pi] \\
 & : & \mathbb{C}(A) \to \texttt{list}\ \mathbb{C}(A) \\
y_x & \equiv & \lambda\pi.\,[\,] \\
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(\lambda y.\,t)_x & \equiv & \lambda(y,\pi).\,t_x\,\pi \\
 & : & \mathbb{W}(A) \times \mathbb{C}(B) \to \texttt{list}\ \mathbb{C}(\Gamma_i) \\
(t\,u)_x & \equiv & \lambda\pi.\,(((\texttt{snd}\ t^\bullet)\,\pi\,u^\bullet) \gg\!= u_x)\ @\ t_x\,(u^\bullet,\pi) \\
 & : & \mathbb{C}(B) \to \texttt{list}\ \mathbb{C}(\Gamma_i)
\end{array}
$$

(We can generalize this interpretation to algebraic datatypes.)

- The standard Dialectica only returns one stack
  - $\rightsquigarrow$ the first correct stack, dynamically tested

Dialectica Reloaded

- The standard Dialectica only returns one stack
  - $\rightsquigarrow$ the first correct stack, dynamically tested
- This is somehow a weak form of delimited control
  - $\rightsquigarrow$ Inspectable stacks: $\sim A$ vs. $\neg A$
  - $\rightsquigarrow$ First class access to those stacks with $(-)_x$
  - $\rightsquigarrow$ Or through a control operator

$$\mathscr{D} : (A \to B) \to A \to \sim B \to \texttt{list}(\sim A)$$

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- We can do the same thing with other calling conventions
  - $\rightsquigarrow$ The protohistoric Dialectica was call-by-name
  - $\rightsquigarrow$ Choose your favorite translation into LL!

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- They have the right multiplicity...
- But they are not respecting the KAM order!
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The faulty one is the application case (more generally duplication).

$$(t\, u)_x \equiv \lambda\pi.\, (((\mathtt{snd}\ t^\bullet)\, \pi\, u^\bullet) \gg= u_x)\ @\ t_x\, (u^\bullet, \pi)$$

A deep issue

- The KAM imposes us sequentiality
- We want to reflect it into the translation

A deep issue

- The KAM imposes us sequentiality
- We want to reflect it into the translation
- Alas, no way to do that
- The $\parr$ translation is far too symmetrical
  - $\leadsto$ We want *interleaving*
  - $\leadsto$ Dialectica can't achieve it as is
  - $\leadsto$ Polarization? Tensorial logic? Dump Dialectica?

I lied (again)

We still did not reach the protohistoric Dialectica.

- To encode MP and IP we need $\varnothing$ as a proof.
  - $\rightsquigarrow$ not only as a stack
  - $\rightsquigarrow$ $\varnothing$ behaves like an exception

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- To encode MP and IP we need $\varnothing$ as a proof.
  - $\rightsquigarrow$ not only as a stack
  - $\rightsquigarrow$ $\varnothing$ behaves like an exception
- In our setting we only get a weak version of MP

$$\widetilde{MP} : \neg(\forall x : A. \sim P[x]) \to (\forall x : A. \sim P[x]) \to \mathfrak{M}\ (\exists x : A. P[x])$$

- And not IP.

Towards CC^ω

- What about more expressive systems?
- We follow the computation intuition we presented
- ... and we apply Dialectica to dependent types
  - $\leadsto$ subsuming first-order logic;
  - $\leadsto$ a proof-relevant $\forall$;
  - $\leadsto$ towards CC^ω and further!

Main lines

- We keep the CBN λ-calculus
  - $\rightsquigarrow$ it can be lifted readily to dependent types
  - $\rightsquigarrow$ $A \to B$ becomes $\Pi x : A.\, B$
  - $\rightsquigarrow$ $A \times B$ becomes $\Sigma x : A.\, B$
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  - $\leadsto$ nothing special to do!
- Design choice: types have no computational content (effect-free):
  - $\leadsto$ a bit disappointing;
  - $\leadsto$ but it works...
  - $\leadsto$ and the usual CC presentation does not help much!

Type translation

Idea: if A is a type,

$$\begin{array}{rcl} A^\bullet \equiv & (\mathbb{W}(A), \mathbb{C}(A)) : \mathtt{Type} \times \mathtt{Type} & \\ A_x \equiv & \lambda\pi.\, [] & \text{(effect-free)} \end{array}$$

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We get:

$$\begin{array}{rcl} \mathtt{Type}^\bullet & \equiv & (\mathtt{Type} \times \mathtt{Type}, 1) \\ \mathtt{Type}_x & \equiv & \lambda\pi.\,[\,] \\ (\Pi y : A.\, B)^\bullet & \equiv & \left(\begin{array}{c} (\Pi y : \mathbb{W}(A).\, \mathbb{W}(B)) \\ \times \\ (\Pi y : \mathbb{W}(A).\, \mathbb{C}(B) \to \mathfrak{M}\ \mathbb{C}(A)) \end{array}, \Sigma y : \mathbb{W}(A).\, \mathbb{C}(B) \right) \\ (\Pi y : A.\, B)_x & \equiv & \lambda\pi.\,[\,] \end{array}$$

Soundness

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- We need finite multisets
  - HITs, HITs, HITs!
- We need some commutative cut rules
  - First class (read: negative) records may do the trick
- Or extensionality hammer
  - Maybe Oury-like tricks

Inductives, please

- We can obtain dependent destruction quite easily

$$\frac{\Gamma \vdash t : A + B \qquad \Gamma, x : A \vdash u_1 : C[\mathtt{L}\ x] \qquad \Gamma, y : B \vdash u_2 : C[\mathtt{R}\ y]}{\Gamma \vdash \mathtt{case}\ t\ \mathtt{with}\ [\mathtt{L}\ x \Rightarrow u_1 \mid \mathtt{R}\ y \Rightarrow u_2] : C[t]}$$

- Just tweak the linear decomposition and there you go!

Conclusion

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- It is an approximation of two side-effects:
  - $\leadsto$ A bit of delimited control (the $(-)_x$ part)
  - $\leadsto$ A form of exceptions (with $\varnothing$)
- But is is partially wrong:
  - $\leadsto$ it is oblivious of sequentiality
  - $\leadsto$ how can we fix it?
- The delimited control part can be lifted seamlessly to CC^ω
  - $\leadsto$ as soon as we have a little bit more than CC
  - $\leadsto$ we need a more computation-relevant presentation of CC

Thanks for your attention.