An Authoritarian Approach to Presheaves

Pierre-Marie Pédrot

INRIA

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It's Time to CIC Ass

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Good Properties We Love

Consistency There is no proof of `False`.

Implementability Type-checking is decidable.

Canonicity Closed integers are indeed integers, i.e

$$\vdash M : \mathbb{N} \quad \text{implies} \quad M \equiv \mathtt{S} \ldots \mathtt{S}\,\mathtt{O}$$

Assuming we have a notion of reduction compatible with conversion:

Normalization Reduction is normalizing

Subject reduction Reduction is compatible with typing

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PRESHEAVES!

- Bread and Butter of Model Construction
- Proof-relevant Kripke semantics
- a.k.a. Intuitionistic Forcing

A Bit of Categorical Nonsense

Definition

Let $\mathbb{P}$ be a category. A presheaf over $\mathbb{P}$ is just a functor $\mathbb{P}^{\mathsf{op}} \to \mathbf{Set}$.

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Bear with me, we will handwave through this in the next slides.

All Your Base Category Are Belong to Us

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- A family of $\mathbb{P}$-indexed sets $\mathbf{A}_p : \mathbf{Set}$
- A family of "restriction morphisms"

$$\theta_{\mathbf{A}} : \Pi\{p, q \in \mathbb{P}\}\,(\alpha \in \mathbb{P}(q, p)).\ \mathbf{A}_p \to \mathbf{A}_q$$

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s.t. given $x \in \mathbf{A}_p$, $\alpha \in \mathbb{P}(q, p)$ and $\beta \in \mathbb{P}(r, q)$:

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"Lowering is compatible with the structure of $\mathbb{P}$"

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- which is natural, i.e. given $x \in \mathbf{A}_p$ and $\alpha \in \mathbb{P}(q, p)$

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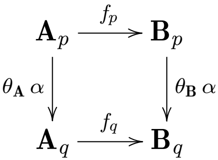
"f is compatible with restriction"



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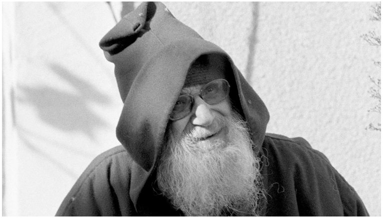

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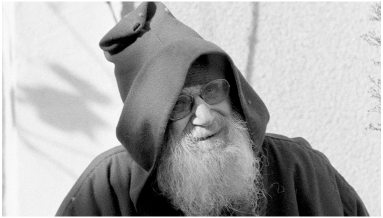

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For our purposes, that means that

- $\mathsf{Psh}(\mathbb{P})$ is some kind of type theory
- ... in particular, it contains the simply-typed λ-calculus

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As usual:

$$\vdash A : \Box \quad \rightsquigarrow \quad [\![A]\!] \in \mathsf{Psh}(\mathbb{P})$$

$$\vdash M : A \quad \rightsquigarrow \quad [M] \in \mathsf{Nat}(1, [\![A]\!])$$

I won't give further details here. One remark though.

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Yet another set-theoretical model!

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Phenomenological Law

Set-theoretical models suck.

Syntactic Models

What is a model?

- Takes syntax as input.
- Interprets it into some low-level language.
- Must preserve the meaning of the source.
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“By Jove, this is a *compiler*!”

This is a folklore in the Curry-Howard community.

On Curry-Howard Poetry

Usual models are more like **interpreters**.

No separation between $\left\{\begin{array}{l}\text{implementation}\\\text{meta}\end{array}\right\}$ vs. $\left\{\begin{array}{l}\text{host}\\\text{target}\end{array}\right\}$ languages

$$\vdash_{\mathcal{S}} A \quad \xrightarrow{\text{meta}} \quad \vDash_{\mathcal{M}} A$$

Notably, $\vDash_{\mathcal{M}}$ lives in the semantical world.

Example: NbE, external realizability.

On Curry-Howard Poetry

Syntactic models are proper **compilers**.

Target and meta languages are **clearly distinct**.

$$\vdash_{\mathcal{S}} A \quad \xrightarrow{\text{meta}} \quad \vdash_{\mathcal{T}} [\![A]\!]$$

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We will be interested in instances where $\mathcal{S}, \mathcal{T}$ are type theories.

Syntactic Models, Details

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Step 3: Expand $\mathcal{S}$ by going down to the $\mathcal{T}$ *assembly language*, implementing new terms through the $[\cdot]$ translation.

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Obviously, that's subtle.

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Yet, a lot of nice consequences.

- Does not require non-type-theoretical foundations (*monism*)
- Can be implemented in Coq (*software monism*)
- Easy to show (relative) consistency, look at $[\![\texttt{False}]\!]$
- Inherit properties from CIC: computationality, decidability, **implementation**...

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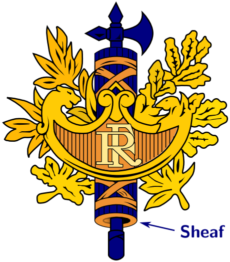
PRESHEAVES!

“Is it possible to see the presheaf construction as a syntactic model?”



French Coat of Arms

Persevere Diabolicum

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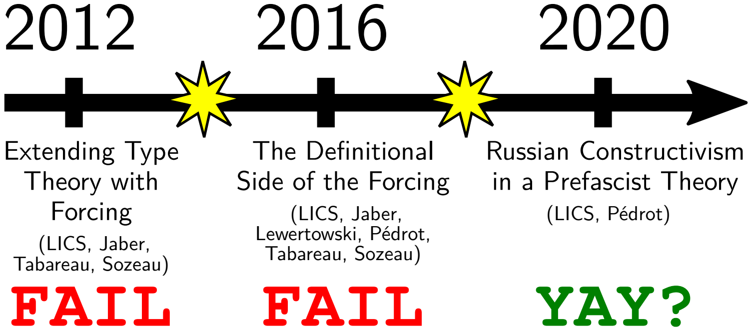

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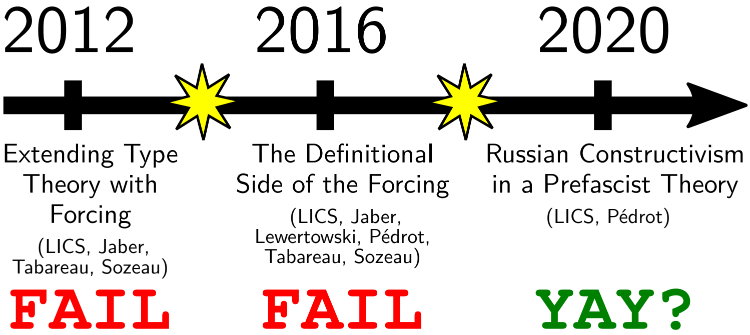

It is the journey, not the destination

2012

(We were warned.)

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What could possibly go wrong?

Close Encounters of the Third Type

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$$\texttt{Cat} : \square \quad := \quad \left\{ \begin{array}{l} \mathbb{P} : \square \\ \leq : \mathbb{P} \rightarrow \mathbb{P} \rightarrow \square \\ \texttt{id} : \Pi p.\, p \leq p \\ \circ : \Pi p\, q\, r.\, p \leq q \rightarrow q \leq r \rightarrow p \leq r \\ \texttt{eqn} : \ldots; \end{array} \right\}$$

$$\texttt{Psh} : \square \quad := \quad \left\{ \begin{array}{l} \mathbf{A} : \mathbb{P} \rightarrow \square \\ \theta_{\mathbf{A}} : \Pi (p\, q : \mathbb{P})\, (\alpha : q \leq p).\, \mathbf{A}_p \rightarrow \mathbf{A}_q \\ \texttt{eqn} : \ldots; \end{array} \right\}$$

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And voilá, the Great Typification is an utter success!

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... except that equations are propositional !!!

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Thus the target theory must be EXTENSIONAL

That Was Not My Intension

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- Arguably better than ZFC (“constructive”)
- ... but undecidable type checking
- ... no computation, e.g. β-reduction is undecidable
- See Théo Winterhalter’s soon to be defended PhD for more horrors

That Was Not My Intension

Extensional Type Theory (ETT) is defined by *Santa Claus conversion*.

$$\frac{\Gamma \vdash e : M = N}{\Gamma \vdash M \equiv N}$$

- Arguably better than ZFC ("constructive")
- ... but undecidable type checking
- ... no computation, e.g. β-reduction is undecidable
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No True Scotsman

Syntactic models into ETT are not really syntactic models†.

That Was Not My Intension

No True Scotsman

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(†) To be more precise, I believe that ETT is not really a type theory.

2016

(Make conversion great again, and break everything else.)

Squaring the Circle

(Me to Guilhem, Nicolas and Matthieu, some time before defending PhD.)

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... and you're trying to intepret a *call-by-name* language!

— What on earth does that even mean?

This is the Left Adjoint, Right?

CBPV is a nice framework to study effects.

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Kripke models factorize through CBPV*.*

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Kripke models factorize through CBPV*.*

$$
\begin{array}{cccc}
X & \text{computation type} & \mapsto & [\![X]\!]^{\mathtt{c}} : |\mathbb{P}| \to \mathbf{Set} \\
A & \text{value type} & \mapsto & [\![A]\!]^{\mathtt{v}} : \mathsf{Fun}(\mathbb{P}^{op}, \mathbf{Set})
\end{array}
$$

$$
\begin{array}{ccl}
[\![A \to X]\!]^{\mathtt{c}}_p & := & [\![A]\!]^{\mathtt{v}}_p \to [\![X]\!]^{\mathtt{c}}_p \\
[\![\mathcal{F}\ A]\!]^{\mathtt{c}}_p & := & |[\![A]\!]^{\mathtt{v}}_p|
\end{array}
$$

$$
[\![\mathcal{U}\ X]\!]^{\mathtt{v}}_p \quad := \quad \Pi(q : \mathbb{P})(\alpha : q \leq p).\, [\![X]\!]^{\mathtt{c}}_q \qquad \text{(free functoriality)}
$$

$$
\theta_{[\![\mathcal{U}\ X]\!]^{\mathtt{v}}}\ (\alpha : q \leq p)(x : [\![\mathcal{U}\ X]\!]^{\mathtt{v}}_p) := \lambda(r : \mathbb{P})(\beta : r \leq q).\, x\ r\ (\alpha \circ \beta)
$$

More Than One Way to Do It

Theorem

Kripke models factorize through CBPV.

Canonical embeddings of λ-calculus into CBPV:

CBN	$(\sigma \to \tau)^{\mathsf{N}}$	$:=$	$\mathcal{U}\,\sigma^{\mathsf{N}} \to \tau^{\mathsf{N}}$	(a computation type)
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Thus, composing the CBV embedding with the "Kripke" interpretation:

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This is the presheaf interpretation of arrows! (up to naturality)**

Le Clash

Presheaves are *call-by-value*!

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In particular, they only satisfy the CBV equational theory generated by

$$(\lambda x.\, t)\ V \equiv_{\beta v} t\{x := V\}$$

because

$$t \equiv_{\beta v} u \quad \longrightarrow \quad t^{\mathsf{V}} \equiv_{\mathsf{CBPV}} u^{\mathsf{V}} \quad \longrightarrow \quad [t^{\mathsf{V}}]_p \equiv_{\mathcal{T}} [u^{\mathsf{V}}]_p$$

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Folklore

Call-by-name is not call-by-value!

If There is No Solution, There is No Problem

Easy solution! Pick the CBN decomposition instead.

$$[\![(\sigma \to \tau)^{\mathsf{N}}]\!]_p^{\mathsf{c}} := (\Pi(q : \mathbb{P})(\alpha : q \leq p).\, [\![\sigma^{\mathsf{N}}]\!]_q^{\mathsf{c}}) \to [\![\tau^{\mathsf{N}}]\!]_p^{\mathsf{c}}$$

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There is a syntactic presheaf model of CC^ω *into* CIC.

where CC^ω is CIC without inductive types.

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$$\begin{array}{rcl}
\vdash_{\mathsf{CC}^\omega} A : \Box & \longrightarrow & p : \mathbb{P} \vdash_{\mathsf{CIC}} [A]_p : \Pi(q : \mathbb{P})(\alpha : q \leq p).\, \Box \\
\vdash_{\mathsf{CC}^\omega} M : A & \longrightarrow & p : \mathbb{P} \vdash_{\mathsf{CIC}} [M]_p : [A]_p\ p\ \mathtt{id}_p \\
\vdash_{\mathsf{CC}^\omega} M \equiv N & \longrightarrow & p : \mathbb{P} \vdash_{\mathsf{CIC}} [M]_p \equiv [N]_p
\end{array}$$

Robbing Peter to Pay Paul

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The model disproves dependent elimination!

in general $\quad \not\vdash \Pi(P : \mathbb{B} \rightarrow \square).\, P\ \mathtt{tt} \rightarrow P\ \mathtt{ff} \rightarrow \Pi(b : \mathbb{B}).\, P\ b$

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because there are **non-standard** booleans.

It only validates it for specific predicates P

$$\vdash P\ \mathtt{tt} \to P\ \mathtt{ff} \to \Pi(b : \mathbb{B}).\, P\ b \qquad \text{if } P \text{ strict}$$

- Any predicate P can be made strict canonically (using storage operators)
- In presence of dep. elim. strictification is the identity

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The Proverbial Paul

CBPV Folklore

- In effectful CBV, functions are not functions. (no substitution)
- In effectful CBN, inductive types are not inductive types. (no dep. elim.)

Conclusion of the Episode II

Good News

This is one of the first reasonable example of dependent effects.

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Bad News

We still don't have a syntactic presheaf model.

Interlude

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In the meantime we worked quite a bit on effectful type theories

- Weaning translation
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This helped us understand what we first missed!

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Categorical presheaves form a model of the whole λ-calculus.

... in particular, it does interpret full β-conversion (although extensionally).

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- We do not have an equivalent in our CBN interpretation
- Isn't this some ad-hoc trick?

Completely Unrelated Slide

Consider an effectful CBV λ-calculus.

Definition (Führmann '99)

A term $t : A$ is said to be **thunkable** if it satisfies the equation

$$\texttt{let}\ x := t\ \texttt{in}\ \lambda().\, x \quad \equiv \quad \lambda().\, t$$

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Theorem (Folklore Realizability)

The sublanguage of hereditarily thunkable terms satisfies full β-conversion.

$$f \Vdash A \to B \quad := \quad \forall u. \quad u \Vdash A \quad \longrightarrow \quad f\ u\ \mathsf{thk} \quad \wedge \quad f\ u \Vdash B$$

Presheaves Are (Pure) Call-By-Value!

Theorem

A term $x : A \vdash t : B$ *is thunkable in the Kripke semantics iff* $[t]_p$ *is natural.*

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Literal unfolding of the definitions. □

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Psh($\mathbb{P}$) is the "pure" subcategory of an effectful CBV language!

- This is a systematic construction.
- Unfortunately it relies on extensionality.
- We *know* how to port this to the CBN setting **intensionally**.

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The CBN equivalent is parametricity!

Syntactic Models For Free

Bernardy-Lasson '11

There is a well-known parametricity interpretation for type theory

$$\Gamma \vdash_{\mathsf{CIC}} M : A \quad \longrightarrow \quad [\![\Gamma]\!]_\varepsilon \vdash_{\mathsf{CIC}} [M]_\varepsilon : [\![A]\!]_\varepsilon\ M$$

where $\quad [\![\cdot]\!]_\varepsilon := \cdot \quad$ and $\quad [\![\Gamma, x : A]\!]_\varepsilon := [\![\Gamma]\!]_\varepsilon, x : A, x_\varepsilon : [\![A]\!]_\varepsilon\ x$

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Given another syntactic model $[-]/[\![-]\!]$ we can define

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Bernardy-Lasson is parametricity over identity.

On Parametric Presheaves

What does parametricity look like on the CBN presheaf model?

$$x : \mathbb{B} \quad \longrightarrow \quad \begin{cases} x : (\Pi(q : \mathbb{P})(\alpha : q \leq p).\, \mathbb{B}) \\ x_\varepsilon : \mathbb{B}_\varepsilon\ p\ x \end{cases}$$

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We have a bit of constraints. To get dependent elimination we need:

1. $\mathbb{B}_\varepsilon\ p\ x$ iff $(x = \lambda q\,\alpha.\,\texttt{tt})$ or $(x = \lambda q\,\alpha.\,\texttt{ff})$

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But we also critically need to be compatible with the presheaf structure!

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😱 Guess what? The CBV vs. CBN conundrum is back. 😱

Trouble All The Way Up

This is exactly the CBV vs. CBN conundrum **one level higher**

Either you pick $\mathbb{B}_\varepsilon\ p\ x := (x = \lambda q\,\alpha.\,\mathtt{tt}) + (x = \lambda q\,\alpha.\,\mathtt{ff})$

$\rightsquigarrow$ this satisfies unicity but breaks definitionality (i.e. CBV).

Or you freeify $\mathbb{B}_\varepsilon\ p\ x := \Pi q\,\alpha.(\alpha \cdot x = \lambda r\,\beta.\,\mathtt{tt}) + (\alpha \cdot x = \lambda r\,\beta.\,\mathtt{ff})$

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Or you freeify $\mathbb{B}_\varepsilon\ p\ x := \Pi q\,\alpha.(\alpha \cdot x = \lambda r\,\beta.\,\mathtt{tt}) + (\alpha \cdot x = \lambda r\,\beta.\,\mathtt{ff})$

$\rightsquigarrow$ this satisfies definitionality but breaks unicity (i.e. CBN).

It is not possible to get both at the same time in CIC!

Playing Cubes

We could solve this with infinite towers of parametricity.

That is, the n-level proof is guaranteed to be pure by then $(n+1)$-level one.

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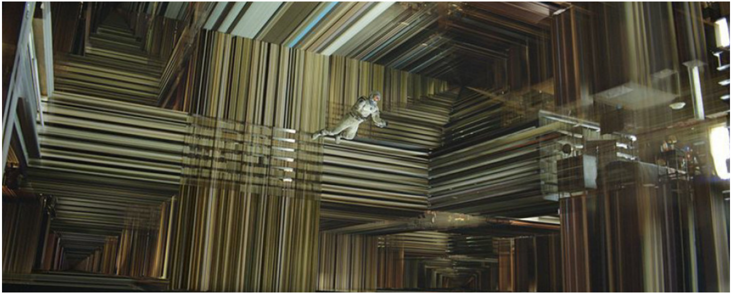

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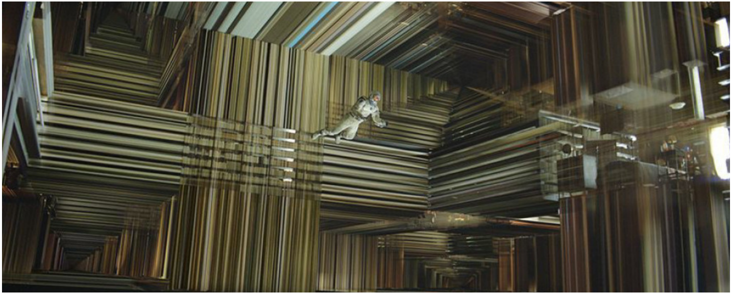

``Oh noes, not cubical type theory again!''

But CuTT itself is justified by presheaf models.

What would be the point to implement presheaves using presheaves?

(On the virtues of Authoritarianism.)

A New Hope

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They introduce a new sort SProp of **strict propositions**.

$$M, N : A : \texttt{SProp} \quad \longrightarrow \quad \vdash M \equiv N$$

- It can be seen as a well-behaved subset of Prop
- It is compatible with HoTT
- It enjoys all good syntactic properties (SN, canonicity, decidability...)
- Coq has it impredicative, Agda has a parallel hierarchy $\texttt{SProp}_i$

Strict Propositions

Critically, `SProp` is closed under products.

$$\vdash A : \Box, \qquad x : A \vdash B : \texttt{SProp} \quad \longrightarrow \quad \vdash \Pi(x : A).\, B : \texttt{SProp}$$

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Accepting the elimination of eq gives rise to a **strict equality**.

A Strict Doctrine

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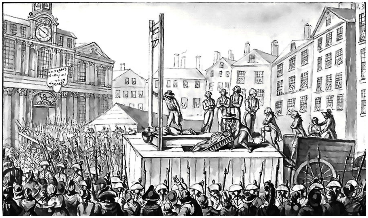
... the authoritarian $\mathfrak{s}$CIC will instead **guillotine** all higher equalities.



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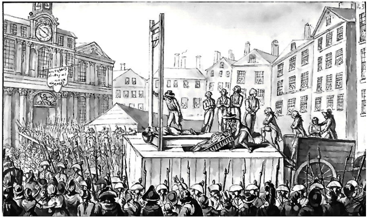

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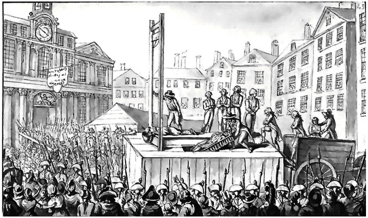

Art. 1. *All humans are born uniquely equal in rights.*

Strict equality is the authoritarian way to solve the coherence hell.

(By default, `SProp` as implemented in Coq doesn't take side, you have to opt-in.)

Strict Parametricity

In the parametric presheaf translation,

- make the parametricity predicate free $\rightsquigarrow$ **definitional functoriality**
- require it to be a strict proposition $\rightsquigarrow$ **proof uniqueness**

$$x : A \quad \longrightarrow \quad \left\{ \begin{array}{l} x : (\Pi(q : \mathbb{P})(\alpha : q \leq p).\, [\![A]\!]_q) \\ x_\varepsilon : (\Pi(q : \mathbb{P})(\alpha : q \leq p).\, [\![A]\!]_\varepsilon\ q\ (\alpha \cdot x)) \end{array} \right.$$

where critically $[\![A]\!]_\varepsilon\ p\ x : \texttt{SProp}$.

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We call the result the **prefascist translation**. (lat. *fascis* : sheaf)

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We call the result the **prefascist translation**. (lat. *fascis* : sheaf)

Theorem (Pédrot '20)

The prefascist translation is a syntactic model of CIC *into* $\mathfrak{s}$CIC.

- Full conversion, full dependent elimination.
- The actual construction is a tad involved, but boils down to the above.
- Unsurprinsingly, UIP is required to interpret universes (tricky!).

No Pain, No Gain

$\mathfrak{s}$CIC is way weaker than ETT

$\mathfrak{s}$CIC is **conjectured** to enjoy the usual good syntactic properties.

- Canonicity seems relatively easy to show
- UIP makes reduction depend on conversion though
- SN is problematic, e.g. $\mathfrak{s}$CIC + an impredicative universe is **not** SN
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We don't rely on impredicativity in the prefascist model

We would inherit the purported good properties $\mathfrak{s}$CIC for free.

Back to Set

Set is a model of $\mathfrak{s}$CIC

Thus, the prefascist model can also be described set-theoretically.

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A prefascist set $\mathcal{A} := (\mathcal{A}_p, (-) \Vdash_p \mathcal{A})$ over a category $\mathbb{P}$ is given by

- a family of sets $\mathcal{A}_p$ for $p \in \mathbb{P}$.
- a family of predicates $(-) \Vdash_p \mathcal{A} \quad \subseteq \quad \text{Cone}_p(\mathcal{A}) := \Pi(q : \mathbb{P})(\alpha : q \leq p).\, \mathcal{A}_q$

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A prefascist morphism f from $\mathcal{A}$ to $\mathcal{B}$ is

- a family of functions $f_p : \mathsf{El}_p\ \mathcal{A} \rightarrow \mathcal{B}_p$
- preserving predicates, i.e.

$$\forall x : \mathsf{El}_p\ \mathcal{A}. \quad \mathsf{app}_p(f, x) \Vdash_p \mathcal{B}$$

where

$$\begin{array}{lcl} \mathsf{El}_p\ \mathcal{A} & := & \{x : \mathsf{Cone}_p(\mathcal{A}) \mid \forall q\, (\alpha : q \leq p).\, (\alpha \cdot x) \Vdash_q \mathcal{A}\} \\ \mathsf{app}_p(f, x) & := & \lambda q\, (\alpha : q \leq p).\, f_q\ (\alpha \cdot x) \end{array}$$

Through The Looking Glass

Theorem

Prefascist sets over $\mathbb{P}$ form a category $\mathbf{Pfs}(\mathbb{P})$ with ***definitional*** *laws.*

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Takeaway: prefascist sets are a better presentation of presheaves

Application

Russian Constructivism

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Proofs are Kleene realizers

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Thus, the principle that puts it apart both from Brouwer **and** Bishop:

Markov's Principle (MP)

$$\forall (f : \mathbb{N} \to \mathbb{B}).\, \neg\neg(\exists n : \mathbb{N}.\, f\, n = \mathtt{tt}) \to \exists n : \mathbb{N}.\, f\, n = \mathtt{tt}$$

- A lot of equivalent statements, e.g. a TM that doesn't loop terminates
- Semi-classical: $\mathbf{HA}^\omega \subsetneq \mathbf{HA}^\omega + \mathsf{MP} \subsetneq \mathbf{PA}^\omega$
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What if we tried to extend CIC with MP through a syntactic model?

Let's look at the realizer

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Proving `mp` $\Vdash$ MP needs MP in the meta-theory!

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We need something else...

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Not one, but at least **two** alternatives!

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- Coquand-Hofmann's syntactic model for $\mathbf{HA}^\omega + \mathrm{MP}$
- Herbelin's direct style proof using static exceptions

$$\begin{aligned}&\texttt{mp}\ (p : \neg\neg(\exists n.\, f\ n = \texttt{tt})) :=\\ &\quad \texttt{try}_\alpha\ \bot_e\ (p\ (\lambda k.\, k\ (\lambda n.\, \texttt{raise}_\alpha\ n)))\ \texttt{with}\ \alpha\ n \mapsto n\end{aligned}$$

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In the remainder, we'll show that

- Coquand-Hofmann's model scales to CIC
- It can be presented as the composition of two translations
- It has the same computational content as Herbelin's proof

High-level view

CH's model is a mix of Kripke semantics and Friedman's A-translation.

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The secret sauce is that the exception type depends on the current p

Pipelining

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Instead, we present our CIC variant synthetically as the composition

$$\mathsf{CIC} \quad \xrightarrow{\mathbf{Exn}} \quad \mathsf{CIC} + \mathcal{E} \quad \xrightarrow{\mathbf{Pfs}} \quad \mathfrak{s}\mathsf{CIC}$$

where

- **Pfs** is the prefascist model described before
- **Exn** is the exceptional model, a CIC-worthy A-translation

Failure is Not an Option

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$$\vdash_{\mathcal{S}} A : \square \quad \longrightarrow \quad \vdash_{\mathcal{T}} [A] := (\llbracket A \rrbracket, [A]_{\varnothing}) : \Sigma A_0 : \square.\, (\mathcal{E} \rightarrow A_0)$$
$$\vdash_{\mathcal{S}} M : A \quad \longrightarrow \quad \vdash_{\mathcal{T}} [M] : \llbracket A \rrbracket$$

Every type $\llbracket A \rrbracket$ comes with its failure function $[A]_{\varnothing} : \mathcal{E} \rightarrow \llbracket A \rrbracket$

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$$[\![\mathbb{B}]\!] := \mathtt{tt}_{\mathcal{E}} : [\![\mathbb{B}]\!] \mid \mathtt{ff}_{\mathcal{E}} : [\![\mathbb{B}]\!] \mid \mathbb{B}_{\varnothing} : \mathcal{E} \rightarrow [\![\mathbb{B}]\!]$$

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Theorem

Provided there is no closed $M : \mathcal{E}$ in the target theory, the source theory enjoys canonicity. In particular, it is consistent.

Somebody Set Up Us The Bomb

We perform the exceptional translation over an **exotic** type of exceptions

$$\mathsf{CIC} \quad \xrightarrow{\textbf{Exn}} \quad \mathsf{CIC} + \mathcal{E} \quad \xrightarrow{\textbf{Pfs}} \quad \mathfrak{s}\mathsf{CIC}$$

$\mathcal{E}$ exists in the prefascist model over $\mathbb{P} := \mathbb{N} \to \mathbb{B}$.

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There is no closed proof of $\mathcal{E}$ in $\mathsf{CIC} + \mathcal{E}$ since

$$\mathcal{E}_p := \Sigma n : \mathbb{N}.\, \texttt{ff} = \texttt{tt} \qquad \text{for } p \text{ constantly } \texttt{ff}$$

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Therefore, the leftmost source theory is consistent.

Realizing MP

We also have a modality in $\mathsf{CIC} + \mathcal{E}$

$$\begin{array}{lcl} \texttt{local} & : & (\mathbb{N} \rightarrow \mathbb{B}) \rightarrow \Box \rightarrow \Box \\ [\texttt{local}\ \varphi\ A]_p & \mathrel{:\stackrel{\sim}{=}} & [A]_{p \wedge \varphi} \end{array}$$

- $\texttt{return} : A \rightarrow \texttt{local}\ \varphi\ A$
- `local` commutes to arrows and positive types
- $\texttt{local}\ \varphi\ \mathcal{E} \quad \cong \quad \mathcal{E} + (\Sigma n : \mathbb{N}.\, \varphi\ n = \texttt{tt})$

Realizing MP

We also have a modality in $\mathsf{CIC} + \mathcal{E}$

$$\begin{array}{lcl} \texttt{local} & : & (\mathbb{N} \to \mathbb{B}) \to \square \to \square \\ [\texttt{local}\ \varphi\ A]_p & :\stackrel{\sim}{=} & [A]_{p \wedge \varphi} \end{array}$$

- $\texttt{return} : A \to \texttt{local}\ \varphi\ A$
- `local` commutes to arrows and positive types
- $\texttt{local}\ \varphi\ \mathcal{E} \quad \cong \quad \mathcal{E} + (\Sigma n : \mathbb{N}.\, \varphi\ n = \texttt{tt})$

To realize MP, we perform intuitionistic symbol pushing in $\mathsf{CIC} + \mathcal{E}$

$$\begin{array}{lcl} [\![\neg\neg(\Sigma n : \mathbb{N}.\, \varphi\ n = \texttt{tt})]\!]_{\mathcal{E}} & \cong & ((\Sigma n : \mathbb{N}.\, \varphi\ n = \texttt{tt}) \to \mathcal{E}) \to \mathcal{E} \\ & \to & \texttt{local}\ \varphi\ (((\Sigma n : \mathbb{N}.\, \varphi\ n = \texttt{tt}) \to \mathcal{E}) \to \mathcal{E}) \\ & \cong & ((\Sigma n : \mathbb{N}.\, \varphi\ n = \texttt{tt}) \to \texttt{local}\ \varphi\ \mathcal{E}) \to \texttt{local}\ \varphi\ \mathcal{E} \\ & \to & \mathcal{E} + (\Sigma n : \mathbb{N}.\, \varphi\ n = \texttt{tt}) \\ & \to & [\![\Sigma n : \mathbb{N}.\, \varphi\ n = \texttt{tt}]\!]_{\mathcal{E}} \end{array}$$

A Computational Analysis of MP

Every time we go under `local` we get new exceptions!

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The structure of the realizer thus follows closely Herbelin's proof.

$$\begin{array}{l} \texttt{mp}\ (p : \neg\neg(\exists n.\, f\ n = \texttt{tt})) := \\ \qquad \texttt{try}_\alpha\ \bot_e\ (p\ (\lambda k.\, k\ (\lambda n.\, \texttt{raise}_\alpha\ n)))\ \texttt{with}\ \alpha\ n \mapsto n \end{array}$$

In particular p can raise exceptions from outside, which is reflected here.

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Thus, Herbelin's proof is the direct style variant of Coquand-Hofmann

Final Digression

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Neutral terms behave as statically bound exceptions

As our model shows, this two techniques are morally equivalent.

This also highlights suspicious ties between delimited continuations and presheaves.

Conclusion

On presheaves:

- Presheaves are a purified sublanguage of a monotonic reader effect
- We have given a better-behaved presentation of presheaves
- It is a syntactic model that relies on strict equality in the target
- Provides for free extensions of CIC with SN, canonicity and the like
- ... assuming $\mathfrak{s}$CIC enjoys this $(\dagger)$

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TODO:

- Implement cubical type theory in this model

Thanks for your attention.