Hurewicz fibrations in elementary toposes

Revised version

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Intro

Question. *What is a model of HoTT? What is a model of HoTT with univalence?*

Remark. We won't address univalence in this talk...

Typoi

Definition. *Let* $\mathbb{C}$ *be a category with finite limits.*

1. *A class of maps* $\mathcal{F} \subseteq \mathbb{C}_1$ *is a class of* fibrations *provided it contains all isos and is stable under composition as well as under base change.*

2. *Let* $\mathcal{F} \subseteq \mathbb{C}_1$ *be a class of* fibrations*. An object* $X \in \mathbb{C}$ *is* $\mathcal{F}$-fibrant *if* $!_X : X \to 1$ *is a fibration.*

3. *Let* $\mathcal{F} \subseteq \mathbb{C}_1$ *be a class of* fibrations*.* $(\mathbb{C}, \mathcal{F})$ *is a* tribe *provided every object* $X \in \mathbb{C}$ *is* $\mathcal{F}$*-fibrant.*

Definition. *Let* $\mathbb{C}$ *be a category with pullbacks,* $f\colon A \to B$ *be a map and* $(E, p) \in \mathbb{C}/A$*. An object* $\prod_f (E, p) = \left(\prod_f (E), \prod_f (p)\right) \in \mathbb{C}/B$ *equipped with a morphism* $\epsilon\colon \prod_f (E) \times_B A \to E$ *is the* product of E along f *if the morphism*

$$\epsilon_! \, f^*(-)\colon \mathbb{C}/B\left((X, u), \prod_f (E, p)\right) \;\to\; \mathbb{C}/B(f^*(X, u), (E, p))$$

is iso for every $(X, u) \in \mathbb{C}/B$*. The map* ϵ *is called* evaluation*.*

Definition. *A tribe* $(\mathbb{C}, \mathcal{F})$ *is* $\prod$ *- closed provided*

i. every fibration $p\colon E \to A$ *has a product along every fibration* $f\colon A \to B$*;*

ii. and $\prod_f (E, p)$ *is again a fibration.*

Example.

1. Any LCCC with all the maps.
2. Any CCC with projections.
3. Small groupoids with Grothendieck fibrations **(Hoffmann-Streicher)** .
4. Kan complexes with Kan fibrations **(Streicher, Voevodsky)** .
5. Type theory terms with display maps **(Gambino-Garner)**

Definition. *Let* $(\mathbb{C}, \mathcal{F})$ *be a tribe.*

1. *A morphism in* $\mathbb{C}$ *is anodyne provided* $c \in {}^{\pitchfork}\mathcal{F}$*.*
2. *Let* $\mathcal{A} \subseteq \mathbb{C}_1$ *be the class of anodyne morphisms. The tribe* $(\mathbb{C}, \mathcal{F})$ *is* homotopical *provided*
   i. $(\mathcal{A}, \mathcal{F})$ *is a factorisation system;*
   ii. *anodyne morphisms are stable under base change along a fibration.*

Remark. (Joyal) We have $(i) \Rightarrow (ii)$ provided the tribe is $\prod$ - closed.

Definition. *Let* $(\mathbb{C}, \mathcal{F})$ *be a homotopical tribe and* $A \in \mathbb{C}$*. A* path object $\mathbf{P}A$ *is given by an* $(\mathcal{A}, \mathcal{F})$*-factorisation of the diagonal* $\Delta\colon A \to A \times A$*. A homotopy with respect to a path object is called* path homotopy*.*

Remark. Let $(\mathbb{C}, \mathcal{F})$ be a homotopical tribe. A path objects exists and can be "lifted" to slices.

Theorem. (Joyal) *Let* $(\mathbb{C}, \mathcal{F})$ *be a homotopical tribe. The path homotopy relation is a congruence on* $\mathbb{C}$*.*

Definition. *A tribe* $(\mathbb{C}, \mathcal{F})$ *is a* typos *provided*

i. it is homotopical and $\prod$ *- closed;*

ii. the product functor $\prod_f$ *preserves the path homotopy relation for every fibration* f*.*

Theorem. (Hoffman-Streicher) *The tribe of small groupoids and Grothendieck fibrations is a typos.*

Theorem. (Awodey-Warren-Voevodsky) *The tribe of Kan complexes and Kan fibrations is a typos.*

Theorem. (Gambino-Garner) *The tribe of type theory terms and display maps is a typos.*

Question. *How about realisability toposes?*

Remark. What follows is a vast generalisation of the material in Jaap van Oosten's seminal paper *Notion of Homotopy for the Effective Topos* (2010).

Intervals

Fix an elementary topos $\mathbb{T}$.

Definition.

1. $X \in \mathbb{T}$ *is* connected *if* $\mathbb{T}(X, 1+1) = \{\mathrm{inl} \circ !_X, \mathrm{inr} \circ !_X\}$.
2. $I \in \mathbb{T}$ *is an* elementary interval *if it is connected and has precisely two distinct global elements* $\partial_0, \partial_1 \colon 1 \to I$.
3. $I_n \stackrel{\text{def.}}{=} \underbrace{I +_{\partial_0,\partial_1} \cdots +_{\partial_0,\partial_1} I}_{n\times}$.

Remark.

1. $I_0 = 1$ and $I_1 = I$.
2. I_n has precisely $n+1$ global elements $\# i_n \colon 1 \to I_n$ corresponding the the injections into the defining wide pushout.
3. There is the obvious linear order on $\mathbb{T}(1, I_n) = \{\# 0, \cdots, \# n\}$.
4. I_n is connected for all $n \in \mathbb{N}$.

Definition. *A* Hurewicz topos *is an elementary topos with* NNO, *equipped with a distinguished elementary interval.*

Remark. In a Hurewicz topos $\mathbb{T}$ coproducts $\coprod_{n\in\mathbb{N}} X^{I_n}$ exist for all $X \in \mathbb{T}$ since general *bounded (co)limits* exist in any topos.

Example. Any Grothendieck topos.

Example. The *effective topos* $\mathbf{Eff}$ where the distinguished elementary interval is the assembly

$$I = (\{0,1\}; E(0) = \{0,1\}, E(1) = \{1,2\})$$

since there is no uniform realizer for the map

$$\begin{array}{rcl} e\colon I & \longrightarrow & 1+1 \\ i & \mapsto & i \end{array}$$

where $i \in \{0, 1\}$. Notice that in $(\mathbf{Eff}, I)$ an object $(X, \approx)$ is connected provided $E(x) \cap E(x') \neq \varnothing$ for all $x, x' \in X$.

Example. Any realizability topos over a PCA.

Question. *Any realisability topos?*

Remark. As pointed out by several people in the audience, classical toposes (e.g. realizability toposes over *Krivine structures*) cannot be Hurewicz since internal Excluded Middle stands in the way...

Paths in Hurewicz toposes

Assume $\mathbb{T}$ is Hurewicz.

Definition.

1. *$s_n \stackrel{\text{def.}}{=} \#0\colon 1 \to I_n$ and $t_n \stackrel{\text{def.}}{=} \#n \to I_n$ are called I_n's* endpoints.
2. *A map $f\colon I_m \to I_n$ is*
   - endpoint-preserving *if $f \circ s_m = s_n$ and $f \circ t_m = t_n$;*
   - order-preserving *if $\#i_m \leqslant \#j_m \Rightarrow f \circ \#i_m \leqslant f \circ \#j_m$.*
3. *A order and endpoint preserving map is called* degeneracy.

Remark. Suppose $k\colon I_m \to I_n$ is a degeneracy. Then k is epi and $m \geqslant n$.

Definition. *Let* $X \in \mathbb{T}$*. The* path object $X^{\langle I \rangle}$ *of* X *is given by*

$$X^{\langle I \rangle} \stackrel{\text{def.}}{=} \coprod_{n \in \mathbb{N}} X^{I_n} / {\sim}$$

where $\sigma \sim \theta$ *if there is a degeneracy* δ *such that* $\sigma = \theta \circ \delta$ *or* $\theta = \sigma \circ \delta$*.*

Remark.

1. Any path $[\sigma] \in X^{\langle I \rangle}$ has a canonical representative of minimal length.

2. For any two paths $[\sigma], [\theta] \in X^{\langle I \rangle}$ there are always representatives of same length.

3. Let $\rho: I_m \rightarrow X$ and $\rho': I_n \rightarrow X$ be representatives of a path $[\sigma] \in X^{\langle I \rangle}$. Then

$$\rho \circ \# m = \rho' \circ \# n$$

   (since ρ and ρ' are related by a degeneracy).

4. In a Hurewicz topos, *path-connectedness* and *connectedness* are equivalent notions.

Definition. *A* constant path *is a path with the path of length* 0 *among it's representatives.*

Notation. *We shall write* $[\sigma_m]$ *if there is a need to insist that the representative's domain is* I_m.

Remark. The source and target maps $s, t\colon X^{\langle I\rangle} \to X$ given by evaluations $s([\sigma_m]) = \sigma_m(0)$ and $t([\sigma_m]) = \sigma_m(m)$ respectively determine an internal graph $X^{\langle I\rangle} \rightrightarrows X$ in $\mathbb{T}$, since representatives differ by a degeneracy.

Notation. *We shall abuse notation and write* $X^{\langle I\rangle}$ *for the internal graph* $X^{\langle I\rangle} \rightrightarrows X$.

Proposition. *The internal graph* $X^{\langle I\rangle}$ *is an internal category with*

i. identity $c\colon X \to X^{\langle I\rangle}$ *given by* $c(x) = \left[1 \xrightarrow{\ulcorner x \urcorner} X\right]$;

ii. composition $(-*-)\colon X^{\langle I\rangle} \times_X X^{\langle I\rangle} \to X^{\langle I\rangle}$ *given by*

$$([\sigma_m] * [\theta_n])(i) = [\texttt{if}\, i \leqslant m \,\texttt{then}\, \sigma_m(i) \,\texttt{else}\, \theta_n(i)]$$

Moreover, there is a contravariant involution $(-)^{\mathrm{rev}}\colon X^{\langle I\rangle} \to X^{\langle I\rangle}$ *which is constant on objects and given by*

$$[\sigma_n]^{\mathrm{rev}}(i) = \sigma_n(n-i)$$

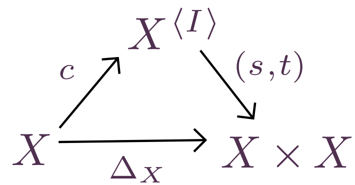
on maps.

Remark. $(-)^{\langle I\rangle}\colon \mathbb{T}\to\mathbb{T}$ is a functor acting on paths by postcomposition:

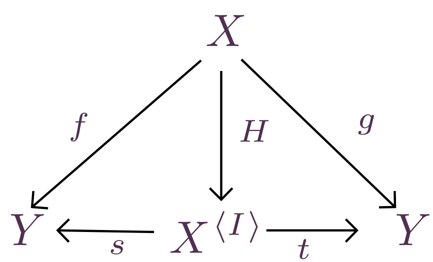
$$f^{\langle I\rangle}([\sigma]) = [f\circ\sigma]$$

Moreover, all the associated maps are natural.

Remark. There is a factorisation of the diagonal map



Definition. *Let $f, g\colon X \to Y$ be maps. A* homotopy $H\colon f \rightsquigarrow g$ *from f to g is given by the commuting diagram*



H *is* constant *on a subobject $X' \lhd X$ if $H(x) = c(x)$ for all $x \in X'$.*

Remark. So for any $x \in X$ we have a path $H(x)\colon f(x) \rightsquigarrow g(x)$.

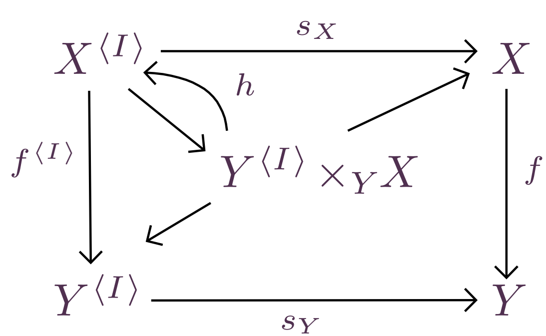
Homotopy equivalences

Definition. *A homotopy equivalence is a map* $u: X \to Y$ *which has an up-to-homotopy inverse* $v: Y \to X$.

Remark. The map v is a homotopy equivalence as well, called *the inverse homotopy equivalence*.

Fibrations

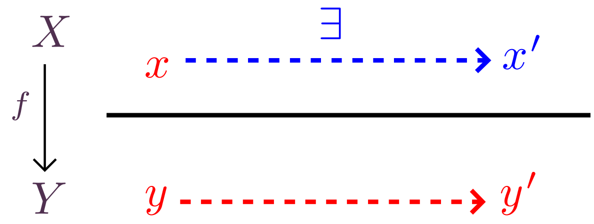
Definition. *A section* h *of the map* $\left(f^{\langle I\rangle}, s_X\right)$



is called Hurewicz connection. *A map which admits a Hurewicz connection is called* Hurewicz fibration.

Notation. *We shall write* $\mathcal{H}$ *for the class of Hurewicz fibrations.*

Remark. A Hurewicz fibration $f\colon X \to Y$ is thus a map with a path lifiting property: for any path $\sigma\colon y \rightsquigarrow y'$ in Y and any $x \in X$ such that $f(x) = y$ there is a path θ in X such that $f \circ \theta = \sigma$:



Definition. $X \in \mathbb{T}$ *is* fibrant *if* $!_X\colon X \to 1$ *is a fibration.*

Proposition.

1. *Fibrations are stable under pullback and composition.*

2. *Any iso is a fibration.*

3. *Any object is fibrant.*

4. $(s, t)\colon X^{\langle I \rangle} \to X$ *is a fibration for any* $X \in \mathbb{T}$.

Strong deformation retracts

Definition. $X \in \mathbb{T}$ *is a* strong deformation retract of $Y \in \mathbb{T}$ *if there is a map* $e\colon X \to Y$ *admitting a retraction* $r\colon Y \to X$ *such that there is a homotopy* $H\colon \mathrm{id}_Y \rightsquigarrow e \circ r$ *constant on* X *(that is* $H(x)\colon x \rightsquigarrow (e \circ r)(x)$ *is the constant path* $c(x)$ *for all* $x \in X$*). The split mono* e *is called* sdr-insertion*.*

Remark.

1. Any sdr-insertion is a homotopy equivalence.
2. Sdr-insertions are stable under pullback along a fibration.

Factorisations

Definition. *A map* $a \in {}^{\pitchfork}\mathcal{H}$ *is called* anodyne*.*

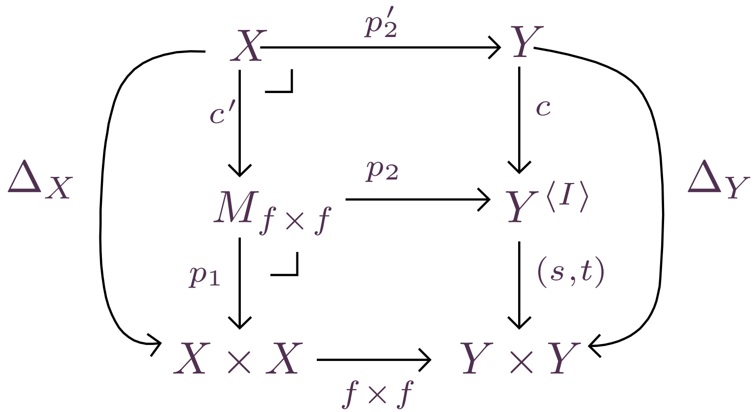
Proposition. *Sdr-insertions are anodyne.*

Definition. *Let* $f\colon X \to Y$ *be a map. The object* M_f *given by the pullback*

$$\begin{array}{ccc} M_f & \xrightarrow{p_2} & Y^{\langle I \rangle} \\ {\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is called f*'s* mapping track*.*

Remark. Pulling back the factorisation $\Delta_Y = (s,t) \circ c$ of the diagonal yields a factorisation of the diagonal $\Delta_X = p_1 \circ c'$ with $p_1 \in \mathcal{H}$ and $c' \in {}^{\pitchfork}\mathcal{H}$:



Theorem. *Any map $f\colon X \to Y$ factors through the mapping track M_f as $f = h \circ a$ with a anodyne and h a Hurewicz fibration.*

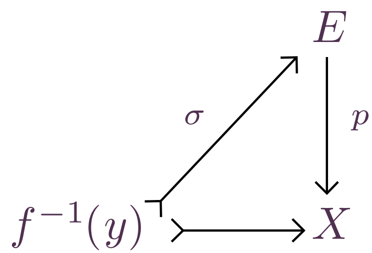
Products along maps

Remark. It is well-known that in any topos $\mathbb{T}$ and any map $f\colon X \to Y$ in $\mathbb{T}$, the pullback functor $f^*\colon \mathbb{T}/Y \to \mathbb{T}/X$ has a left and a right adjoint $\sum_f \dashv f^* \dashv \prod_f$ called *pushforward along* f and *product along* f respectively. The product along f at u is the *object of local sections* of u, that is

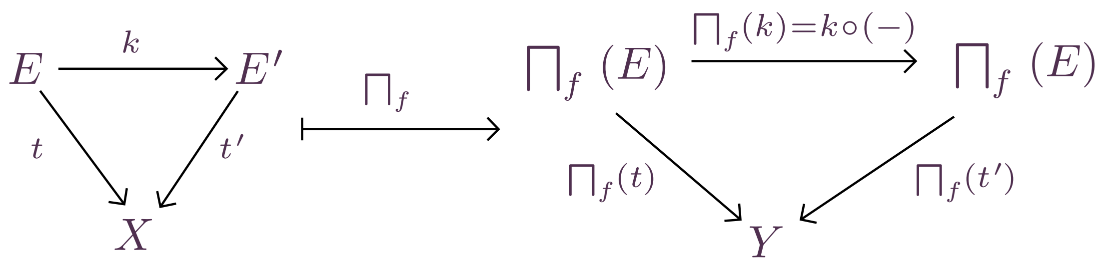
$$\prod_f (X \xrightarrow{u} Y) = \{\sigma \in E^{f^{-1}(y)} \mid y \in Y, p \circ \sigma = i\}$$

in $\mathbb{T}$'s internal language.

Remark. A local section is thus given by the diagram



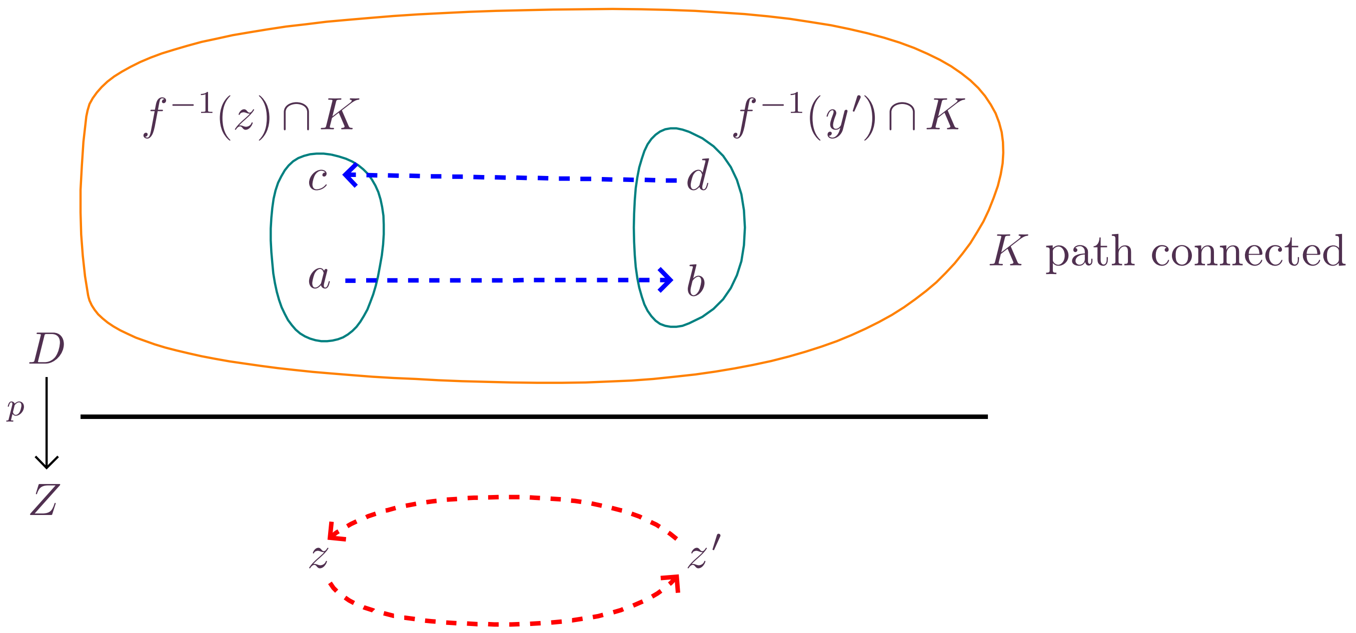
The action on maps is given by postcomposition



Lemma. *Assume $\mathbb{T}$ Hurewicz. Let $p\colon D \to Z$ be a fibration, $K \subseteq D$ a path connected component of D and $\delta\colon z \rightsquigarrow z'$ a path in X. The following are equivalent*

i. $f^{-1}(z) \cap K \neq \varnothing$;

ii. $f^{-1}(z') \cap K \neq \varnothing$

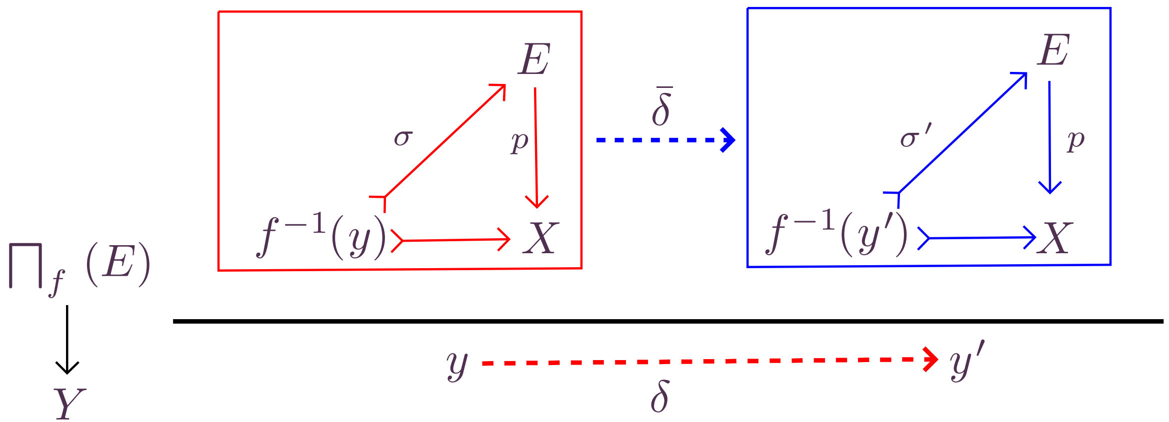


Theorem. *Let* $p, f \in \mathbb{T}_1$ *be fibrations. Then* $\prod_f (p)$ *is again a fibration.*

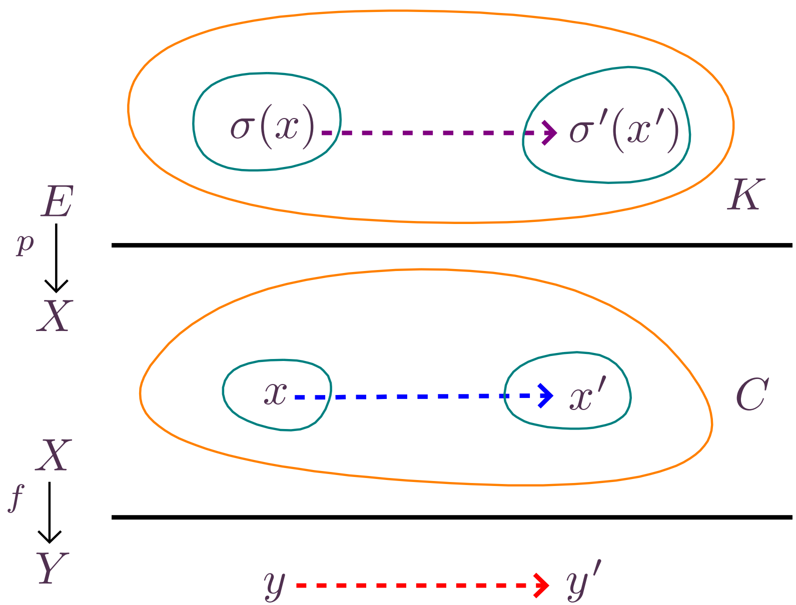
Definition. *Assume* $X \in \mathbb{T}$*.* X*'s* local category *is the subcategory* $L_{\mathcal{F}}(X) \subseteq \mathbb{T}/X$ *of the slice* $\mathbb{T}/X$ *with all the objects being fibrations.*

Corollary. $\prod_f$ *restricts to local categories.*

A path $\delta \colon y \rightsquigarrow y'$ in Y can be lifted to a path of sections:



since we have



The Hurewicz typos

Let $\mathbb{C}$ be a category with finite limits. Recall that the product over A is given by pullback

$$(E,u)\times(E',u')=(E\times_A E',u*u')$$

with $u*u'=u\circ p_1=u'\circ p_2$. The diagonal $\Delta_A E\colon(E,u)\to(E,u)\times(E,u)$ over A is thus given by

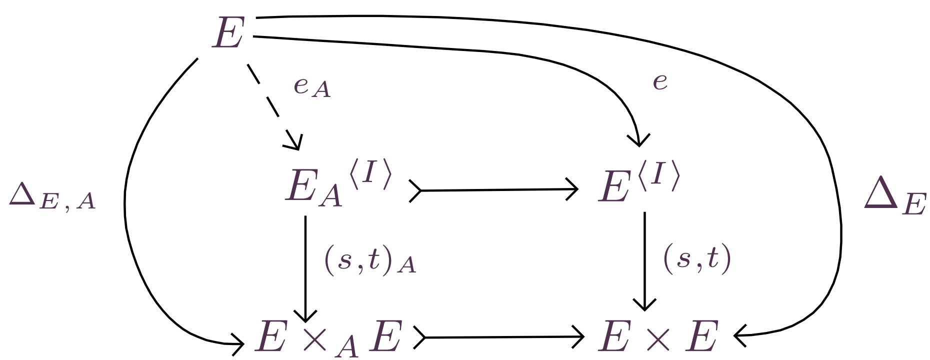
$$\Delta_{E,A}=(u*u)\circ(\mathrm{id}_E,\mathrm{id}_E)_A$$

Definition. *Let $(\mathbb{J}, \mathcal{F})$ be a tribe. A fibration in $\mathbb{J}/A$ is in $\mathcal{F}$ (as a map over A).*

Proposition. (Joyal) *Let $(\mathbb{J}, \mathcal{F})$ be a tribe. The factorisation of the diagonal*

$$\Delta_E = (s, t) \circ e$$

with e anodyne and (s, t) a fibration induces a factorisation in $L_{\mathcal{F}}(A)$ by pullback in $\mathbb{J}$



The path object is $\left(E_A{}^{\langle I \rangle}, (u * u) \circ (s, t)_A\right)$. We have $s_A = p_1 \circ (u * u)$ and $t_A = p_2 \circ (u * u)$.

Lemma. *Let* $\mathbb{T}$ *be a Hurewicz topos and* $f \colon A \to B$*. The functorial square*

$$\begin{array}{ccc} L_{\mathcal{H}}(A) & \xrightarrow{\prod_f} & L_{\mathcal{H}}(B) \\ {\scriptstyle (-)_A{}^{\langle I \rangle}} \downarrow & & \downarrow {\scriptstyle (-)_B{}^{\langle I \rangle}} \\ L_{\mathcal{H}}(A) & \xrightarrow[\prod_f]{} & L_{\mathcal{H}}(B) \end{array}$$

commutes.

Remark. This is much stronger than just preservation of the homotopy relation.

Theorem. *Any Hurewicz topos is a typos.*