Implicative algebras: a new foundation for forcing and realizability

Alexandre Miquel

UNIVERSIDAD DE LA REPUBLICA URUGUAY

FACULTAD DE INGENIERIA

EQUIPO DE LÓGICA UDELAR

July 21th, 2016 – Piriápolis

Introduction

- Krivine's classical realizability is a complete reformulation of Kleene realizability that takes into account classical reasoning
  - Based on Griffin '90 discovery:

$$\text{call/cc} \;:\; ((\phi \Rightarrow \psi) \Rightarrow \phi) \Rightarrow \phi \qquad \text{(Peirce's law)}$$

  - New models for PA2 and ZF (+ DC) [Krivine 03, 09, 12]
- Many connections between classical realizability and Cohen forcing
  - Combination of classical realizability and forcing + Generalization to classical realizability algebras [Krivine 11, 12]
  - Computational analysis of Cohen forcing [M. 11]
- Fascinating model-theoretic perspectives [Krivine 12, 15]
  - Classical realizability = non commutative forcing ?
- **This talk:** An attempt to define a simple algebraic structure that subsumes both forcing and intuitionistic/classical realizability

The significance of classical realizability

- **Tarski models:** $[\![\phi]\!] \in \{0;1\}$
  - Interprets classical provability (correctness/completeness)
- **Intuitionistic realizability:** $[\![\phi]\!] \in \mathfrak{P}(\Lambda)$ [Kleene 45]
  - Interprets intuitionistic proofs
  - Independence results in intuitionistic theories
  - Definitely incompatible with classical logic
- **Cohen forcing:** $[\![\phi]\!] \in \mathfrak{P}(C)$ [Cohen 63]
  - Independence results, in classical theories
    (Negation of continuum hypothesis, Solovay's axiom, etc.)
- **Boolean-valued models:** $[\![\phi]\!] \in \mathcal{B}$ [Scott, Solovay, Vopěnka]
- **Classical realizability:** $[\![\phi]\!] \in \mathfrak{P}(\Lambda_c)$ [Krivine 94, 01, 03, 09, 11–]
  - Interprets classical proofs
  - Generalizes Tarski models... and forcing!

Heyting/Boolean-valued model vs realizability

- In Boolean/Heyting-valued models (or forcing):
  - conjunction interpreted as a meet/intersection...
  - universal quantification interpreted as an infinitary intersection
    $\rightsquigarrow$ amounts to an infinitary conjunction
- In intuitionistic/classical realizability:
  - conjunction interpreted as a Cartesian product
  - universal quantification interpreted as an infinitary intersection

	$\wedge = \forall = \cap$	$\wedge = \times, \forall = \cap$
Int. logic	Heyting-valued models (Kripke forcing)	Int. realizability
Class. logic	Boolean-valued models (Cohen forcing)	Class. realizability

Plan

Plan

Implicative structures

Definition (Implicative structure)

An implicative structure is a triple $(\mathscr{A}, \preccurlyeq, \to)$ where

(1) $(\mathscr{A}, \preccurlyeq)$ is a complete (meet semi-)lattice

(2) $(\to) : \mathscr{A}^2 \to \mathscr{A}$ is a binary operation such that:

(2a) $a' \preccurlyeq a, \quad b \preccurlyeq b' \quad \text{entails} \quad (a \to b) \preccurlyeq (a' \to b') \qquad (a, a', b, b' \in \mathscr{A})$

(2b) $\bigcurlywedge_{b \in B}(a \to b) \;=\; a \to \bigcurlywedge_{b \in B} b \qquad (\text{for all } B \subseteq \mathscr{A})$

- Write $\bot$ (resp. $\top$) the smallest (resp. largest) element of $\mathscr{A}$
- When $B = \varnothing$, axiom (2b) gives: $(a \to \top) = \top \qquad (a \in \mathscr{A})$

Examples of implicative structures (1/2)

Complete Heyting algebras

- Recall that a Heyting algebra is a bounded lattice $(H, \preccurlyeq)$ that has relative pseudo-complements

$$a \to b \;:=\; \max\{c \in H : (c \curlywedge a) \preccurlyeq b\} \qquad \text{(Heyting's implication)}$$

for all $a, b \in H$, so that we get the adjunction:

$$(c \curlywedge a) \preccurlyeq b \;\Leftrightarrow\; c \preccurlyeq (a \to b) \qquad \text{(Heyting's adjunction)}$$

Heyting algebras are models of the intuitionistic propositional calculus. Boolean algebras are the "classical" Heyting algebras, in which $\neg\neg a = a$ for all $a \in H$

- When a Heyting algebra $(H, \preccurlyeq)$ is complete (i.e. has all infinitary meets and joins), it induces an implicative structure $(H, \preccurlyeq, \to)$

Complete Boolean algebras (as a particular case of compl. Heyting algebras)

Examples of implicative structures (2/2)

Total combinatory algebras

- Each total combinatory algebra $(P, \cdot, \mathtt{k}, \mathtt{s})$ induces an implicative structure $(\mathscr{A}, \preccurlyeq, \to)$ defined by
  - $\mathscr{A} \;:=\; \mathfrak{P}(P)$ (sets of combinators)
  - $a \preccurlyeq b \;:=\; a \subseteq b$ (inclusion)
  - $a \to b \;:=\; \{z \in P : \forall x \in a,\ z \cdot x \in b\}$ (Kleene's implication)

  When application is partial, we only get a quasi-implicative structure (cf next slide)

Abstract Krivine structures

- Each abstract Krivine structure (AKS)

$$(\mathbf{\Lambda}, \mathbf{\Pi}, \perp\!\!\!\perp, @, \mathrm{push}, \mathrm{store}, \mathtt{K}, \mathtt{S}, \mathtt{cc}, \mathrm{PL})$$

  induces an implicative structure $(\mathscr{A}, \preccurlyeq, \to)$ defined by:
  - $\mathscr{A} \;:=\; \mathfrak{P}(\mathbf{\Pi})$ (sets of stacks)
  - $a \preccurlyeq b \;:=\; a \supseteq b$ (reverse inclusion)
  - $a \to b \;:=\; a^{\perp\!\!\!\perp} \cdot b$ (Krivine's implication)

Relaxing the definition

In some situations, it is desirable to have $(a \to \top) \neq \top$

Definition (Quasi-implicative structure)

Same definition as for an implicative structure, but axiom

(2b) $$\bigcurlywedge_{b \in B} (a \to b) \;=\; a \to \bigcurlywedge_{b \in B} b \qquad (\text{if } B \neq \varnothing)$$

only required for the non-empty subsets $B \subseteq \mathscr{A}$

Examples:

- Each partial combinatory algebra $(P, \cdot, \mathtt{k}, \mathtt{s})$ more generally induces a quasi-implicative structure: $(\mathfrak{P}(P), \subseteq, \to)$

  This structure is an implicative structure iff application $\cdot$ is total

- Usual notions of reducibility candidates (Tait, Girard, Parigot, etc.) induce quasi-implicative structures (built from the λ-calculus)

Viewing truth values as (generalized) realizers (1/2)

- The Curry-Howard correspondence:

Syntax:	Proof = Program	:	Formula = Type
Semantics:	Realizer	$\in$	Truth value

- But in most semantics, we can associate to every realizer t its principal type $[t]$, i.e. the smallest truth value containing t:

$$t \ : \ A \ \ \text{(typing)} \qquad \text{iff} \qquad [t] \subseteq A \ \ \text{(subtyping)}$$

- Identifying t with $[t]$, we get the inclusion:

$$\text{Realizers} \quad \subset \quad \text{Truth values}$$

- Moreover, we shall see that application and abstraction can be lifted at the level of truth values. Therefore:

$$\text{Truth values} \quad = \quad \text{Generalized realizers}$$

Viewing truth values as (generalized) realizers (2/2)

- Fundamental ideas underlying implicative structures:
  1. Operations on λ-terms can be lifted to truth values
  2. Truth values can be used as generalized realizers
  3. Realizers and truth values live in the same world!

$$\text{Proof} = \text{Program} = \text{Type} = \text{Formula}$$

  (The ultimate Curry-Howard identification)

- In an implicative structure, the relation $a \preccurlyeq b$ may read:
  - a is a subtype of b (viewing a and b as truth values)
  - a has type b (viewing a as a realizer, b as a truth value)
  - a is more defined than b (viewing a and b as realizers)

- In particular:

$$\text{ordering of sybtyping } \preccurlyeq \quad \equiv \quad \text{reverse Scott ordering } \sqsupseteq$$

Encoding application

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ be an implicative structure

Definition (Application)

Given $a, b \in \mathscr{A}$, we let: $\quad ab \ := \ \bigwedge\{c \in \mathscr{A} \ : \ a \preccurlyeq (b \to c)\}$

- From the point of view of the Scott ordering:

$$ab \ := \ \bigsqcup\{c \in \mathscr{A} \ : \ (b \to c) \sqsubseteq a\}$$

- **Properties:**
  1. If $a \preccurlyeq a'$ and $b \preccurlyeq b'$, then $ab \preccurlyeq a'b'$ (Monotonicity)
  2. $(a \to b)a \preccurlyeq b$ ("β-reduction")
  3. $a \preccurlyeq (b \to ab)$ ("η-expansion")
  4. $ab = \min\{c \in a : a \preccurlyeq (b \to c)\}$ (Minimum)
  5. $ab \preccurlyeq c \quad \text{iff} \quad a \preccurlyeq (b \to c)$ (Adjunction)

Encoding abstraction

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ be an implicative structure

Definition (Abstraction)

Given $f : \mathscr{A} \to \mathscr{A}$, we let: $\quad \boldsymbol{\lambda} f \ := \ \bigcurlywedge_{a \in \mathscr{A}} (a \to f(a))$

- From the point of view of the Scott ordering:

$$\boldsymbol{\lambda} f \ := \ \bigsqcup_{a \in \mathscr{A}} (a \to f(a))$$

- **Properties:**

  1. If $f \preccurlyeq g$ (pointwise), then $\boldsymbol{\lambda} f \preccurlyeq \boldsymbol{\lambda} g$ (Monotonicity)
  2. $(\boldsymbol{\lambda} f) a \ \preccurlyeq \ f(a)$ (β-reduction)
  3. $a \ \preccurlyeq \ \boldsymbol{\lambda}(x \mapsto ax)$ (η-expansion)

Encoding the λ-calculus

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ be an implicative structure

- To each closed λ-term t with parameters (i.e. constants) in $\mathscr{A}$, we associate a truth value $t^{\mathscr{A}} \in \mathscr{A}$:

$$\begin{array}{rcl} a^{\mathscr{A}} & := & a \\ (\lambda x \,.\, t)^{\mathscr{A}} & := & \boldsymbol{\lambda}(a \mapsto (t\{x := a\})^{\mathscr{A}}) \\ (tu)^{\mathscr{A}} & := & t^{\mathscr{A}} u^{\mathscr{A}} \end{array}$$

- **Properties:**
  - β-rule: If $t \twoheadrightarrow_\beta t'$, then $(t)^{\mathscr{A}} \preccurlyeq (t')^{\mathscr{A}}$
  - η-rule: If $t \twoheadrightarrow_\eta t'$, then $(t)^{\mathscr{A}} \succcurlyeq (t')^{\mathscr{A}}$

- **Remarks:**
  - This is *not* a denotational model of the λ-calculus!
  - The map $t^{\mathscr{A}}$ is not injective in general

Semantic typing (1/2)

Elements of $\mathscr{A}$ can be used as semantic types for λ-terms:

- **Types:** $a \in \mathscr{A}$

  Terms: λ-terms with parameters in $\mathscr{A}$

  Contexts: $\Gamma \equiv x_1 : a_1, \ldots, x_n : a_n \qquad (a_1, \ldots, a_n \in A)$

  Judgment: $\Gamma \vdash t : a$

- **Remark:** Each context $\Gamma \equiv x_1 : a_1, \ldots, x_n : a_n$ can also be used as a substitution: $\Gamma \equiv x_1 := a_1, \ldots, x_n := a_n$

- The validity of a judgment is defined directly (i.e. semantically); not from a set of inference rules:

Definition (Semantic validity)

$$\Gamma \vdash t : a \quad :\equiv \quad FV(t) \subseteq \mathrm{dom}(\Gamma) \ \text{ and } \ (t[\Gamma])^{\mathscr{A}} \preccurlyeq a$$

Semantic typing (2/2)

Definition (Semantic validity)

$$\Gamma \vdash t : a \qquad :\equiv \qquad FV(t) \subseteq \mathrm{dom}(\Gamma) \ \text{ and } \ (t[\Gamma])^{\mathscr{A}} \preccurlyeq a$$

Proposition

The following semantic typing rules are valid:

$$\frac{}{\Gamma \vdash x : a}\ ((x{:}a)\in\Gamma) \qquad \frac{}{\Gamma \vdash a : a} \qquad \frac{}{\Gamma \vdash t : \top}\ (FV(t)\subseteq \mathrm{dom}(\Gamma))$$

$$\frac{\Gamma, x : a \vdash t : b}{\Gamma \vdash \lambda x\,.\,t : a \to b} \qquad \frac{\Gamma \vdash t : a \to b \quad \Gamma \vdash u : a}{\Gamma \vdash tu : b}$$

$$\frac{\Gamma \vdash t : a_i \quad (\text{for all } i\in I)}{\Gamma \vdash t : \bigcurlywedge_{i\in I} a_i} \qquad \frac{\Gamma \vdash t : a}{\Gamma \vdash t : a'}\ (a\preccurlyeq a') \qquad \frac{\Gamma \vdash t : a}{\Gamma' \vdash t : a}\ (\Gamma'\preccurlyeq\Gamma)$$

Note: $\Gamma' \preccurlyeq \Gamma$ means: $\Gamma'(x) \preccurlyeq \Gamma(x)$ for all $x \in \mathrm{dom}(\Gamma) \subseteq \mathrm{dom}(\Gamma')$.

Remarkable identities (1/2)

- Recall that in (Curry-style) system F, we have:

$$\begin{array}{lcll}
\mathbf{I} & := & \lambda x\,.\,x & : \ \forall\alpha\,(\alpha\to\alpha)\\
\mathbf{K} & := & \lambda xy\,.\,x & : \ \forall\alpha,\beta\,(\alpha\to\beta\to\alpha)\\
\mathbf{S} & := & \lambda xyz\,.\,xz(yz) & : \ \forall\alpha,\beta,\gamma\,((\alpha\to\beta\to\gamma)\to(\alpha\to\beta)\to\alpha\to\gamma)
\end{array}$$

Proposition

In any implicative structure $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ we have:

$$\begin{array}{lcll}
\mathbf{I}^{\mathscr{A}} & := & (\lambda x\,.\,x)^{\mathscr{A}} & = \ \bigcurlywedge_{a}(a\to a)\\
\mathbf{K}^{\mathscr{A}} & := & (\lambda xy\,.\,x)^{\mathscr{A}} & = \ \bigcurlywedge_{a,b}(a\to b\to a)\\
\mathbf{S}^{\mathscr{A}} & := & (\lambda xyz\,.\,xz(yz))^{\mathscr{A}} & = \ \bigcurlywedge_{a,b,c}((a\to b\to c)\to(a\to b)\to a\to c)
\end{array}$$

Remarkable identities (2/2)

- The same property holds for:

$$\mathbf{C} \;:=\; \lambda xyz\,.\,xzy \;:\; \forall\alpha,\beta,\gamma\,((\alpha\to\beta\to\gamma)\to\beta\to\alpha\to\gamma)$$
$$\mathbf{W} \;:=\; \lambda xy\,.\,xyy \;:\; \forall\alpha,\beta\,((\alpha\to\alpha\to\beta)\to\alpha\to\beta)$$

but not for

$$\mathbf{II} \;:=\; (\lambda x\,.\,x)(\lambda x\,.\,x) \;:\; \forall\alpha\,(\alpha\to\alpha)$$

(Thanks to a remark of Étienne Miquey)

- By analogy, we let:

$$\text{cc}^{\mathscr{A}} \;:=\; \bigcurlywedge_{a,b}(((a\to b)\to a)\to a) \qquad \text{(Peirce's law)}$$
$$=\; \bigcurlywedge_{a}((\neg a\to a)\to a) \qquad \text{(where } \neg a := (a\to\bot))$$

From this, we extend the encoding of the λ-calculus to all λ-terms enriched with the constant cc (= proof-like λ_c-terms)

Particular case: $\mathscr{A}$ is a complete Heyting algebra (1/2)

Complete Heyting algebras are the particular implicative structures $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ where $\to$ is defined from the ordering $\preccurlyeq$ by

$$a \to b \ := \ \max\{c \in \mathscr{A} \ : \ (c \curlywedge a) \preccurlyeq b\}$$

Remark: Complete Heyting (or Boolean) algebras are the structures underlying forcing (in the sense of Kripke or Cohen)

Proposition

When $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ is a complete Heyting algebra:

1. For all $a, b \in \mathscr{A}$: $ab = a \curlywedge b$ (application = binary meet)
2. For all λ-terms t with free variables $x_1, \ldots, x_k$ $(k \geq 0)$ and for all $a_1, \ldots, a_k \in \mathscr{A}$, we have:
$$(t\{x := a_1, \ldots, x := a_k\})^{\mathscr{A}} \succcurlyeq a_1 \curlywedge \cdots \curlywedge a_k$$
3. In particular, when t is closed: $(t)^{\mathscr{A}} = \top$
4. $\mathscr{A}$ is a (complete) Boolean algebra iff $\text{cc}^{\mathscr{A}} = \top$

Particular case: $\mathscr{A}$ is a complete Heyting algebra (2/2)

Proof.

1. For all $c \in \mathscr{A}$, we have: $ab \preccurlyeq c \quad \Leftrightarrow \quad a \preccurlyeq (b \to c) \quad \Leftrightarrow \quad a \curlywedge b \preccurlyeq c$, hence $ab = a \curlywedge b$.

2. We prove that $(t\{\vec{x} := \vec{a}\})^{\mathscr{A}} \succcurlyeq a_1 \curlywedge \cdots \curlywedge a_k$ by induction on t
   - $t \equiv x$ (variable). Obvious.
   - $t \equiv t_1 t_2$ (application). Obvious from point 1.
   - $t \equiv \lambda x_0 \,.\, t_0$ (abstraction). In this case, we have:

$$\begin{aligned} (t\{\vec{x} := \vec{a}\})^{\mathscr{A}} &= \bigcurlywedge_{a_0} \big(a_0 \to (t_0\{x_0 := a_0, \vec{x} := \vec{a}\})^{\mathscr{A}}\big) \\ &\succcurlyeq \bigcurlywedge_{a_0} \big(a_0 \to a_0 \curlywedge a_1 \curlywedge \cdots \curlywedge a_k\big) \qquad \text{(by IH)} \\ &\succcurlyeq a_1 \curlywedge \cdots \curlywedge a_k \end{aligned}$$

   using the relation $b \preccurlyeq (a \to a \curlywedge b)$ of Heyting Algebras.

3. In particular, when t is closed, we get: $(t)^{\mathscr{A}} \succcurlyeq \top$

4. $(\mathscr{A}, \preccurlyeq)$ Boolean algebra iff $\text{œ}^{\mathscr{A}} = \top$: Obvious. □

Logical strength of an implicative structure

- **Warning!** We may have $(t)^{\mathscr{A}} = \bot$ for some closed λ-term t.

  Intuitively, this means that the corresponding term is inconsistent in (the logic represented by) the implicative structure $\mathscr{A}$

- We say that the implicative structure $\mathscr{A}$ is:
  - intuitionistically consistent when $(t)^{\mathscr{A}} \neq \bot$ for all closed λ-terms
  - classically consistent when $(t)^{\mathscr{A}} \neq \bot$ for all closed λ-terms with $\mathbf{c}\!\!\!\mathbf{c}$

- **Examples:**
  - Every non-degenerated complete Heyting algebra is int. consistent
  - Every non-degenerated complete Boolean algebra is class. consistent
  - Every implicative structure induced by a total combinatory algebra is intuitionistically consistent
  - Every implicative structure induced by an AKS whose pole $\perp\!\!\!\perp$ is coherent (cf [Krivine'12]) is classically consistent

Two trivial examples...

Trivial example 1:

- Given a complete lattice $(\mathscr{A}, \preccurlyeq)$, we let
$$a \to b \ := \ b \qquad \text{(for all } a, b \in \mathscr{A}\text{)}$$
Clearly, $(\mathscr{A}, \preccurlyeq, \to)$ is an implicative structure
- In this structure, we have: $\mathbf{I}^{\mathscr{A}} \ := \ \bigcurlywedge_a (a \to a) \ = \ \bigcurlywedge_a a \ = \ \bot \quad (!)$

Trivial example 2:

- Given a complete lattice $(\mathscr{A}, \preccurlyeq)$, we let
$$a \to b \ := \ \top \qquad \text{(for all } a, b \in \mathscr{A}\text{)}$$
Again, $(\mathscr{A}, \preccurlyeq, \to)$ is an implicative structure!
- In this structure, we have: $\mathbf{I}^{\mathscr{A}} \ := \ \bigcurlywedge_a (a \to a) \ = \ \top,$ but
$$(\mathbf{II})^{\mathscr{A}} \ := \ \top\top \ = \ \bigcurlywedge\{c \in \mathscr{A} : \top \preccurlyeq (\top \to c)\} \ = \ \bigcurlywedge \mathscr{A} \ = \ \bot \quad (!)$$

… and a non trivial example (1/2)

(The following example is inspired from Girard's phase semantics for LL)

- Let $(M, \cdot, 1)$ be a commutative monoid. We let:
  - $\mathscr{A} \;:=\; \mathfrak{P}(M)$
  - $a \preccurlyeq b \;:=\; a \subseteq b$
  - $a \to b \;:=\; \{\gamma \in M : (\forall \alpha \in a)\, \gamma\alpha \in b\}$ (for all $a, b \in \mathscr{A}$)

  Clearly, $(\mathscr{A}, \preccurlyeq, \to)$ is an implicative structure
  (since the product $\cdot$ is a total operation)

- We easily check that for all $a, b \in \mathscr{A}$:

$$ab \;:=\; a \cdot b \;=\; \{\alpha\beta : \alpha \in a,\ \beta \in b\}$$

  Therefore:
  - $ab = ba$ (application is commutative)
  - $(ab)c = a(bc)$ (application is associative)
  - $aa \neq a$, in general (application is not idempotent)

... and a non trivial example (2/2)

Proposition

1. In the implicative structure $(\mathscr{A}, \preccurlyeq, \to) = (\mathfrak{P}(M), \subseteq, \to)$:

$$\begin{array}{rcl} \mathbf{I}^{\mathscr{A}} &:=& (\lambda x\,.\,x)^{\mathscr{A}} &=& \{1\} &\neq& \bot \\ \mathbf{C}^{\mathscr{A}} &:=& (\lambda xyz\,.\,xzy)^{\mathscr{A}} &=& \{1\} &\neq& \bot \\ \mathbf{B}^{\mathscr{A}} &:=& (\lambda xyz\,.\,x(yz))^{\mathscr{A}} &=& \{1\} &\neq& \bot \end{array}$$

2. Moreover, if we assume that $\alpha^2 \neq \alpha$ for some $\alpha \in M$, then:

$$\begin{array}{rcl} \mathbf{K}^{\mathscr{A}} &:=& (\lambda xy\,.\,x)^{\mathscr{A}} &=& \varnothing &=& \bot \\ \mathbf{W}^{\mathscr{A}} &:=& (\lambda xy\,.\,xyy)^{\mathscr{A}} &=& \varnothing &=& \bot \\ \mathbf{S}^{\mathscr{A}} &:=& (\lambda xyz\,.\,xz(yz))^{\mathscr{A}} &=& \varnothing &=& \bot \end{array}$$

More generally, for each closed λ-term t, we (should) have:

$$(t)^{\mathscr{A}} = \begin{cases} \{1\} & \text{if } t \text{ is linear} \\ \varnothing & \text{otherwise} \end{cases} \qquad \text{(to be checked)}$$

Plan

Separators

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ be an implicative structure

Definition (Separator)

A separator of $\mathscr{A}$ is a subset $S \subseteq \mathscr{A}$ such that:

(1) If $a \in S$ and $a \preccurlyeq b$, then $b \in S$ (upwards closed)

(2) $\mathbf{K}^{\mathscr{A}} = (\lambda xy\,.\,x)^{\mathscr{A}} \in S$ and $\mathbf{S}^{\mathscr{A}} = (\lambda xyz\,.\,xz(yz))^{\mathscr{A}} \in S$

(3) If $(a \to b) \in S$ and $a \in S$, then $b \in S$ (modus ponens)

We say that S is consistent (resp. classical) when $\bot \notin S$ (resp. $\text{\ae}^{\mathscr{A}} \in S$)

Remarks:

- Under (1), axiom (3) is equivalent to:

  (3′) If $a, b \in S$, then $ab \in S$ (closure under application)

- In a complete Heyting algebra: separator = filter
- But in general, separators are not closed under binary meets

λ-terms and separators

Intuition: Separator $S \subseteq \mathscr{A}$ = criterion of truth (in $\mathscr{A}$)

- All separators are closed under the operations of the λ-calculus:

Proposition

Given a separator $S \subseteq \mathscr{A}$:

1. For all λ-terms t with free variables $x_1, \ldots, x_k$ and for all $a_1, \ldots, a_k \in S$, we have: $(t\{x_1 := a_1, \ldots, x_k := a_k\})^{\mathscr{A}} \in S$
2. For all closed λ-terms t: $(t)^{\mathscr{A}} \in S$

- Alternative formulation:

Given a closed λ-term t with parameters in S:

$$\vdash t : a \quad \text{implies} \quad a \in S$$

If a has a "proof" t (possibly using "axioms" $\in S$), then a is true ($\in S$)

Intuitionistic and classical cores

Definition (intuitionistic & classical cores)

Given an implicative algebra we write:

- $S_J^0(\mathscr{A})$ the smallest separator of $\mathscr{A}$ (intuitionistic core)
- $S_K^0(\mathscr{A})$ the smallest classical separator of $\mathscr{A}$ (classical core)

We easily check that:

$$S_J^0(\mathscr{A}) \quad = \quad \uparrow\{(t)^{\mathscr{A}} : t \text{ closed } \lambda\text{-term}\}$$
$$S_K^0(\mathscr{A}) \quad = \quad \uparrow\{(t)^{\mathscr{A}} : t \text{ closed } \lambda\text{-term with } \text{œ}\}$$

writing $\uparrow B$ the upwards closure of a subset $B \subseteq \mathscr{A}$

Proposition

An implicative algebra $\mathscr{A}$ is intuitionistically (resp. classically) consistent if and only if $\bot \notin S_J^0(\mathscr{A})$ (resp. $\bot \notin S_K^0(\mathscr{A})$)

Encoding conjunction and disjunction

In any implicative structure, conjunction and disjunction are defined by:

$$a \times b \quad := \quad \bigcurlywedge_{c} \big((a \to b \to c) \to c\big) \qquad \text{(conjunction)}$$

$$a + b \quad := \quad \bigcurlywedge_{c} \big((a \to c) \to (b \to c) \to c\big) \qquad \text{(disjunction)}$$

Proposition

The following semantic typing rules are valid:

$$\frac{\Gamma \vdash t : a \quad \Gamma \vdash u : b}{\Gamma \vdash \lambda z\,.\,z\,t\,u : a \times b} \qquad \frac{\Gamma \vdash t : a \times b}{\Gamma \vdash t\,(\lambda xy\,.\,x) : a} \qquad \frac{\Gamma \vdash t : a \times b}{\Gamma \vdash t\,(\lambda xy\,.\,y) : b}$$

$$\frac{\Gamma \vdash t : a}{\Gamma \vdash \lambda zw\,.\,z\,t : a + b} \qquad \frac{\Gamma \vdash t : b}{\Gamma \vdash \lambda zw\,.\,w\,t : a + b}$$

$$\frac{\Gamma \vdash t : a + b \quad \Gamma, x : a \vdash u : c \quad \Gamma, y : b \vdash v : c}{\Gamma \vdash t\,(\lambda x\,.\,u)\,(\lambda y\,.\,v) : c}$$

Moreover, we have: $(\lambda z\,.\,z\,a\,b)^{\mathscr{A}} = \langle a, b\rangle^{\mathscr{A}} = a \times b$ (pairing = conjunction)

Encoding quantifiers

Given a family $(a_i)_{i\in I}$, we let:

$$\bigvee_{i\in I} a_i \quad := \quad \bigcurlywedge_{i\in I} a_i$$

$$\exists_{i\in I} a_i \quad := \quad \bigcurlywedge_{c\in\mathscr{A}}\Big(\bigcurlywedge_{i\in I}(a_i \to c) \;\to c\Big)$$

Proposition

The following semantic typing rules are valid:

$$\frac{\Gamma \vdash t : a_i \quad (\text{for all } i\in I)}{\Gamma \vdash t : \forall_{i\in I} a_i} \qquad \frac{\Gamma \vdash t : \forall_{i\in I} a_i}{\Gamma \vdash t : a_{i_0}}\ (i_0\in I)$$

$$\frac{\Gamma \vdash t : a_{i_0}}{\Gamma \vdash \lambda z\,.\,z\,t : \exists_{i\in I} a_i}\ (i_0\in I) \qquad \frac{\Gamma \vdash t : \exists_{i\in I} a_i \qquad \Gamma, x : a_i \vdash u : c \quad (\text{for all } i\in I)}{\Gamma \vdash t\,(\lambda x\,.\,u) : c}$$

Note: The simpler encoding $\exists_{i\in I} a_i \;:=\; \curlyvee_{i\in I} a_i$ does not work in classical realizability

Interpreting 1st-order logic

Definition (Interpretation of a 1st-order language in $\mathscr{A}$)

An interpretation of a 1st-order language $\mathscr{L}$ in $\mathscr{A}$ is given by

- an interpretation $[\![\cdot]\!]$ of 1st-order terms in some set $M \neq \varnothing$
- a function $[\![p]\!] : M^k \to \mathscr{A}$ for each k-ary predicate symbol p

Each formula ϕ of $\mathscr{L}$ (with a valuation ρ) is interpreted in $\mathscr{A}$ by:

$$[\![p(t_1,\ldots,t_n)]\!]_\rho = [\![p]\!]([\![t_1]\!]_\rho,\ldots,[\![t_k]\!]_\rho)$$

$$[\![\phi \Rightarrow \psi]\!]_\rho = [\![\phi]\!]_\rho \to [\![\psi]\!]_\rho \qquad [\![\neg\phi]\!]_\rho = [\![\phi]\!]_\rho \to \bot$$

$$[\![\phi \wedge \psi]\!]_\rho = [\![\phi]\!]_\rho \times [\![\psi]\!]_\rho \qquad [\![\phi \vee \psi]\!]_\rho = [\![\phi]\!]_\rho + [\![\psi]\!]_\rho$$

$$[\![\forall x\,\phi]\!]_\rho = \forall_{v\in M}[\![\phi]\!]_{\rho,x\leftarrow v} \qquad [\![\exists x\,\phi]\!]_\rho = \exists_{v\in M}[\![\phi]\!]_{\rho,x\leftarrow v}$$

Theorem (Soundness)

If ϕ is an intuitionistic (resp. classical) tautology, then:

$$[\![\phi]\!]_\rho \in S_J^0(\mathscr{A}) \qquad (\text{resp. } [\![\phi]\!]_\rho \in S_K^0(\mathscr{A}))$$

Remark: The above construction easily extends to 2nd-/higher-order logic

Implicative algebras

- Given an interpretation $[\![\cdot]\!]$ of a 1st-order language $\mathscr{L}$ in $\mathscr{A}$, each separator $S \subseteq \mathscr{A}$ induces a theory $\mathscr{T}_S$ defined by:

$$\mathscr{T}_S \ := \ \{\phi \text{ closed } : \ [\![\phi]\!] \in S\}$$

  - The larger the separator S, the larger the theory $\mathscr{T}_S$
  - The theory $\mathscr{T}_S$ is consistent iff $\bot_{\mathscr{A}} \notin S$

Definition (Implicative algebra)

An implicative algebra is a quadruple $(\mathscr{A}, \preccurlyeq, \to, S)$ where

- $(\mathscr{A}, \preccurlyeq, \to)$ is an implicative structure
- $S \subseteq \mathscr{A}$ is a separator

The implicative algebra $(\mathscr{A}, \preccurlyeq, \to, S)$ is

- consistent when $\bot \notin S$
- classical when $\text{cc}^{\mathscr{A}} \in S$

Entailment

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to, S)$ be an implicative algebra

- The separator $S \subseteq \mathscr{A}$ induces a relation of entailment

$$a \vdash_S b \ \equiv \ (a \to b) \in S \qquad\qquad (\text{for all } a, b \in \mathscr{A})$$

- The relation $a \vdash_S b$ is clearly a preorder on $\mathscr{A}$, whose corresponding equivalence relation $\dashv\vdash_S$ is given by:

$$\begin{aligned} a \dashv\vdash_S b \ &\equiv \ (a \to b) \in S \text{ and } (b \to a) \in S \\ &\Leftrightarrow \ (a \to b) \times (b \to a) \in S \end{aligned}$$

- In the quotient $\mathscr{A}/S \ := \ \mathscr{A}/{\dashv\vdash_S}$, the preorder $\vdash_S$ induces an order $\leq_S$ defined by

$$[a] \leq_S [b] \ \equiv \ a \vdash_S b$$

(Writing $[a]$ the equivalence class of a modulo S)

The induced Heyting algebra

Proposition

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to, S)$ be an implicative algebra

1. The quotient poset $H = (\mathscr{A}/S, \leq_S)$ is a Heyting algebra, where:

$$\begin{array}{rcl} [a] \to_H [b] & = & [a \to b] \\ [a] \wedge_H [b] & = & [a \times b] \qquad [a] \vee_H [b] \;=\; [a + b] \\ \bot_H & = & [\bot] \qquad \top_H \;=\; [\top] \;=\; S \end{array}$$

2. When $\mathscr{A}$ is classical (i.e. $œ^{\mathscr{A}} \in S$), this poset is a Boolean algebra

The poset $H = (\mathscr{A}/S, \leq_S)$ is called the Heyting algebra induced by $\mathscr{A}$

Remarks:

- The Heyting algebra H is in general not complete
- **Beware!** The ordering $\leq_S$ on H comes from $\vdash_S$ (entailment), and not from $\preccurlyeq$ (subtyping). However, we have: $a \preccurlyeq b \;\Rightarrow\; [a] \leq_S [b]$.

Maximal separators (1/2)

Although separators are *not* filters (w.r.t. the order $\preccurlyeq$), they can be manipulated similarly to filters. For instance:

- We call a maximal separator any separator $S \subseteq \mathscr{A}$ that is consistent and maximal (w.r.t. inclusion) among consistent separators
- By Zorn's lemma, we easily check that any consistent separator can be extended into a maximal separator

Trivial Boolean algebra

$S \subseteq \mathscr{A}$ is a maximal separator if and only if the induced Heyting algebra $(\mathscr{A}/S, \leq_S)$ is the trivial Boolean algebra:

$$S \subseteq \mathscr{A} \text{ maximal} \qquad \text{iff} \qquad (\mathscr{A}/S, \leq_S) \approx \mathbf{2}$$

Remark: Works even when the maximal separator $S \subseteq \mathscr{A}$ is not classical!

Maximal separators (2/2)

Remark: There are non-classical maximal separators!

Typical example is given by intuitionistic realizability:

- Let $(\mathscr{A}, \preccurlyeq, \to)$ be the implicative structure induced by a total combinatory algebra $(P, \cdot, \mathtt{k}, \mathtt{s})$:
  - $\mathscr{A} := \mathfrak{P}(P)$ (sets of combinators)
  - $a \preccurlyeq b := a \subseteq b$ (inclusion)
  - $a \to b := \{z \in P : \forall x \in a,\ z \cdot x \in b\}$ (Kleene's implication)
- Let $S = \mathfrak{P}(P) \setminus \{\varnothing\} = \mathscr{A} \setminus \{\bot\}$. We easily check that S is a consistent separator, obviously maximal. Hence: $\mathscr{A}/S \approx \mathbf{2}$.
- Identity $\mathscr{A}/S \approx \mathbf{2}$ reflects the fact that in intuitionistic realizability, we have either $\Vdash \phi$ or $\Vdash \neg\phi$ for each closed formula ϕ.
- On the other hand, we have: $\text{œ}^{\mathscr{A}} = \bigwedge_a ((\neg a \to a) \to a) = \varnothing$

  (Indeed, from a realizer $t \in \text{œ}^{\mathscr{A}}$, we would easily solve the halting problem)

Separators and filters

- In the theory of implicative algebras, separators play the same role as filters in the theory of Heyting algebras.

  However, separators $S \subseteq \mathscr{A}$ are in general *not* filters:

$$
\begin{array}{rcl}
a, b \in S & \Rightarrow & ab \in S \\
a, b \in S & \Rightarrow & a \times b \in S \\
a, b \in S & \not\Rightarrow & a \curlywedge b \in S
\end{array}
$$

- On the other hand, in the particular case where $\mathscr{A}$ is (derived from) a complete Heyting algebra, we have: separator = filter

- We shall now study in the general case the situations where a separator happens to be also a filter

Non deterministic choice

- Given an implicative structure $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$, we let:

$$⋔^{\mathscr{A}} \;:=\; \bigcurlywedge_{a,b}(a \to b \to a \curlywedge b) \qquad \text{(non deterministic choice)}$$

  We shall also use the symbol ⋔ (non-deterministic choice operator) as an extra constant of the λ-calculus (like cc), that is interpreted by $⋔^{\mathscr{A}}$

- In the λ_c-calculus, universal realizers of the "type" $⋔^{\mathscr{A}}$ are the instructions ⋔ with the non-deterministic evaluation rule:

$$⋔ \star u \cdot v \cdot \pi \;\succ\; \begin{cases} u \star \pi \\ v \star \pi \end{cases} \qquad \text{[Guillermo \& M., 2014]}$$

Non deterministic choice and parallel 'or'

- Let $\quad \mathrm{Nat}^{\mathscr{A}}(n) \;:=\; \bigcurlywedge_{a\in\mathscr{A}^{\mathbb{N}}}\Big(a(0) \to \bigcurlywedge_{p\in\mathbb{N}}\big(a(p)\to a(p+1)\big) \to a(n)\Big)$

Fact

1. $\pitchfork^{\mathscr{A}} \;=\; (\lambda xy\,.\,x)^{\mathscr{A}} \curlywedge (\lambda xy\,.\,y)^{\mathscr{A}}$ $\qquad$ ($\mathtt{tt} \curlywedge \mathtt{ff}$)
2. $\pitchfork^{\mathscr{A}} \;\dashv\vdash_S\; \bigcurlywedge_{n\in\mathbb{N}} \mathrm{Nat}^{\mathscr{A}}(n)$ $\qquad$ (in any separator $S \subseteq \mathscr{A}$)

- Non deterministic choice is related to the parallel 'or'

$$\text{p-or}^{\mathscr{A}} \;:=\; (\bot \to \top \to \bot) \curlywedge (\top \to \bot \to \bot) \qquad \text{(parallel `or')}$$

Fact

1. $\pitchfork^{\mathscr{A}} \;\preccurlyeq\; \text{p-or}^{\mathscr{A}}$
2. $\pitchfork^{\mathscr{A}} \;\dashv\vdash_S\; \text{p-or}^{\mathscr{A}}$ $\qquad$ (in any classical separator $S \subseteq \mathscr{A}$)

Non deterministic choice, parallel 'or' and filters

- Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ be an implicative structure
- It is clear that a separator $S \subseteq \mathscr{A}$ is a filter if and only if it is closed under binary meets: $a, b \in S \;\Rightarrow\; a \curlywedge b \in S$ (for all $a, b \in \mathscr{A}$)

Proposition (Characterizing filters)

1. A separator $S \subseteq \mathscr{A}$ is a filter if and only if: $\pitchfork^{\mathscr{A}} \in S$
2. A classical separator $S \subseteq \mathscr{A}$ is a filter if and only if: $\text{p-or}^{\mathscr{A}} \in S$

Proof.

1. ($\Rightarrow$) In any separator $S \subseteq \mathscr{A}$, we have $(\lambda xy\,.\,x)^{\mathscr{A}}, (\lambda xy\,.\,y)^{\mathscr{A}} \in S$. So that when S is a filter, we get $\pitchfork^{\mathscr{A}} = (\lambda xy\,.\,x)^{\mathscr{A}} \curlywedge (\lambda xy\,.\,y)^{\mathscr{A}} \in S$.

   ($\Leftarrow$) If $\pitchfork^{\mathscr{A}} \in S$, then $(a \to b \to a \curlywedge b) \in S$ for all $a, b \in \mathscr{A}$. So that if $a, b \in S$, we get $a \curlywedge b$ (applying the modus ponens twice in S).
2. Obvious from item 1, since: $\pitchfork^{\mathscr{A}} \in S$ iff $\text{p-or}^{\mathscr{A}} \in S$. □

Generating separators

- Given any subset $X \subseteq \mathscr{A}$, we write:
  - $\mathrm{App}(X)$ the applicative algebra generated by X, i.e. the smallest subset of $\mathscr{A}$ containing X and closed under application
  - $\uparrow X$ the upwards closure of X in $\mathscr{A}$ (w.r.t. $\preccurlyeq$)

Lemma (Separator generated by a subset of $\mathscr{A}$)

For all $X \subseteq \mathscr{A}$, the subset $\uparrow\mathrm{App}(X \cup \{\mathbf{K}^{\mathscr{A}}, \mathbf{S}^{\mathscr{A}}\}) \subseteq \mathscr{A}$ is the smallest separator of $\mathscr{A}$ containing X as a subset

- A separator $S \subseteq \mathscr{A}$ is finitely generated when it is of the form

$$S = \uparrow\mathrm{App}(X) \qquad \text{for some finite subset } X \subseteq \mathscr{A}$$

- We observe that both separators $S_J^0(\mathscr{A}) \subseteq \mathscr{A}$ (intuitionistic core) and $S_K^0(\mathscr{A}) \subseteq \mathscr{A}$ (classical core) are finitely generated

Finitely generated separators and principal filters (1/4)

Theorem

Given a separator $S \subseteq \mathscr{A}$, the following are equivalent:

1. S is finitely generated and $⋔^{\mathscr{A}} \in S$
2. S is a principal filter: $\quad S \;=\; \uparrow\{\Theta\}\quad$ for some $\Theta \in S$

   (Θ is called the universal proof of S)
3. The induced Heyting algebra $H := (\mathscr{A}/S, \leq_S)$ is complete, and the surjection $[\,\cdot\,] : \mathscr{A} \to H$ commutes with infinitary meets:

$$\Big[\bigcurlywedge_{i\in I} a_i\Big] \;=\; \bigwedge_{i\in I}[a_i]$$

In model theoretic terms, this situation corresponds to a collapse of (intuitionistic/classical) realizability into (Kripke/Cohen) forcing!

Finitely generated separators and principal filters (2/4)

Proof.

- S finitely generated + $\pitchfork^{\mathscr{A}} \in \mathscr{S} \quad \Rightarrow \quad S$ principal filter

Suppose that $S = \uparrow\mathrm{App}(\{g_1, g_2, \ldots, g_n\})$ is a filter. Since S is a filter, we have $\pitchfork^{\mathscr{A}} := \bigcurlywedge_{a,b}(a \to b \to a \curlywedge b) \in S$, and more generally:

$$\begin{aligned}\pitchfork_k^{\mathscr{A}} \;&:=\; \bigcurlywedge_{a_1,\ldots,a_k} (a_1 \to \cdots \to a_k \to a_1 \curlywedge \cdots \curlywedge a_k) \\ &=\; \bigcurlywedge_{i=1..k} (\lambda x_1 \cdots x_k \,.\, x_i)^{\mathscr{A}} \;\in\; S\end{aligned}$$

for all $k \geq 1$. We let: $\Theta \;:=\; (\mathbf{Y}\,(\lambda r\,.\,\pitchfork_{n+1}^{\mathscr{A}}\, g_1 \,\cdots\, g_n\,(r\,r)))^{\mathscr{A}} \;\in\; S$

where $\mathbf{Y} \equiv (\lambda yf\,.\,f(yyf))(\lambda yf\,.\,f(yyf))$ is Turing's fixpoint combinator.

By construction we have $\Theta \;\preccurlyeq\; \pitchfork_{n+1}^{\mathscr{A}}\, g_1 \,\cdots\, g_n\,(\Theta\,\Theta)$, hence:

$$\Theta \preccurlyeq g_1, \quad \ldots, \quad \Theta \preccurlyeq g_n \quad \text{and} \quad \Theta \preccurlyeq \Theta\,\Theta$$

By induction, we get $\Theta \preccurlyeq a$ for all $a \in \mathrm{App}(g_1, \ldots, g_n)$, and thus $\Theta \preccurlyeq a$ for all $a \in S$. Therefore: $\Theta = \min(S)$ and $S = \uparrow\{\Theta\}$. (...)

Finitely generated separators and principal filters (3/4)

Proof (continued).

- S principal filter $\Rightarrow$ H complete + commutation property

  Suppose that $S = \uparrow\{\Theta\}$, and let $[a_i]_{i\in I} \in H^I$ be a family of elements of H, defined from a family of representatives $(a_i)_{i\in I} \in \mathscr{A}^I$. Since $\left(\curlywedge_{i\in I} a_i\right) \preccurlyeq a_i$ for all $i \in I$, $\left[\curlywedge_{i\in I} a_i\right]$ is a lower bound of the family $[a_i]_{i\in I}$ in H.

  Conversely, if $[b]$ is a lower bound of the family $[a_i]_{i\in I}$ in H, we have $(b \to a_i) \in S$ for all $i \in I$. And since $S = \uparrow\{\Theta\}$, we get $\Theta \preccurlyeq (b \to a_i)$ for all $i \in I$, so that:

  $$\Theta \;\preccurlyeq\; \curlywedge_{i\in I}(b \to a_i) \;=\; b \to \curlywedge_{i\in I} a_i .$$

  Hence $[b] \leq_S \left[\curlywedge_{i\in I} a_i\right]$. Therefore, $\left[\curlywedge_{i\in I} a_i\right]$ is the g.l.b. of the family $[a_i]_{i\in I}$, hence the commutation property $\left[\curlywedge_{i\in I} a_i\right] = \bigwedge_{i\in I}[a_i]$. (...)

Finitely generated separators and principal filters (4/4)

Proof (continued).

- H complete + commut. property $\Rightarrow$ S finitely generated + $\pitchfork^{\mathscr{A}} \in S$

  Suppose that $H = \mathscr{A}/S$ is complete and that the surjection $[\cdot] : \mathscr{A} \to H$ commutes with infinitary meets. Let $\Theta = \curlywedge S$. From the commutation property, we have:

  $$[\Theta] = \Big[\curlywedge_{a\in S} a\Big] = \bigwedge_{a\in S}[a] = \bigwedge_{a\in S} \top_H = \top_H,$$

  hence $\Theta \in S$, so that $\Theta = \min(S)$ and $S = \uparrow\{\Theta\}$. Therefore the separator S is a (principal) filter, hence we have $\pitchfork^{\mathscr{A}} \in S$.

  S is also finitely generated, by the unique generator Θ. □

Uniform existential quantification

- We say that an implicative structure $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ has uniform existential quantification when for all $(a_i)_{i \in I} \in \mathscr{A}^I$ and $b \in \mathscr{A}$:

$$(*) \qquad \bigcurlywedge_{i \in I}(a_i \to b) \;=\; \Big(\bigcurlyvee_{i \in I} a_i\Big) \to b$$

- This equality (that corresponds to $\exists$-elim) holds in:
  - all complete Heyting/Boolean algebras
  - all the implicative algebras induced by total combinatory algebras $(P, \cdot, \mathtt{k}, \mathtt{s})$ (intuitionistic realizability)
- When $(*)$ holds, we can let: $\displaystyle\exists_{i \in I} a_i \;:=\; \bigcurlyvee_{i \in I} a_i$

Proposition

If $\mathscr{A}$ has uniform existential quantifications, then:

1. p-or $:= \; (\bot \to \top \to \bot) \curlywedge (\top \to \bot \to \bot) \;=\; \top$
2. All classical separators $S \subseteq \mathscr{A}$ are filters

Morality: Uniform $\exists/\forall$ (both) are incompatible with classical realizability

Plan

Road map

- We now want to prove that each implicative algebra $(\mathscr{A}, \preccurlyeq, \to, S)$ induces a tripos $\quad \mathsf{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$
  - Recall that **HA** is the category of Heyting algebras
  - Intuitively: tripos = categorical model of higher-order logic
- We already know how to construct similar triposes from:
  - Complete Heyting/Boolean algebras (forcing triposes)
  - Partial combinatory algebras (realizability triposes)
  - Abstract Krivine structures (AKS) [Streicher'12]

  Our aim is thus to subsume all the above constructions
- Triposes are based on the notion of first-order hyperdoctrine, which is the categorical formulation of first-order theories (or models)

Preliminaries: Galois connections and adjoints

- A Galois connection between two posets A and B is a pair of functions $F : A \to B$ and $G : B \to A$ such that:

$$F(x) \leq y \quad \Leftrightarrow \quad x \leq G(y) \qquad \text{(for all } x \in A,\ y \in B)$$

- In this situation (notation: $F \dashv G$), we observe that:
  1. $F : A \to B$ and $G : B \to A$ are necessarily monotonic
  2. $F : A \to B$ is uniquely determined by $G : B \to A$:

     $$F(x) \ = \ \min\{y \in B : x \leq G(y)\} \qquad \text{(for all } x \in A)$$

     F is called the left adjoint of G, and written $F = G_L$
  3. $G : B \to A$ is uniquely determined by $F : A \to B$:

     $$G(y) \ = \ \max\{x \in A : F(x) \leq y\} \qquad \text{(for all } y \in B)$$

     G is called the right adjoint of F, and written $G = F_R$

Preliminaries: the category **HA** of Heyting algebras

Morphisms of Heyting algebras

Given two Heyting algebras H, H', a function $F : H \to H'$ is a morphism of Heyting algebras when for all $x, y \in H$:

$$\begin{aligned} F(x \wedge y) &= F(x) \wedge F(y) & \qquad F(\top) &= \top \\ F(x \vee y) &= F(x) \vee F(y) & \qquad F(\bot) &= \bot \\ F(x \to y) &= F(x) \to F(y) \end{aligned}$$

A morphism of Heyting algebras is thus a morphism of bounded lattices that also preserves Heyting's implication

- In what follows, we shall mainly consider morphisms of Heyting algebras $F : H \to H'$ with left & right adjoints $F_L, F_R : H' \to H$
  - When they exist, both adjoints are monotonic and unique, but they are in general *not* morphisms of Heyting algebras
  - Note that each isomorphism $F : H \to H'$ has left and right adjoints:
    $$F_L = F_R = F^{-1}$$

Preliminaries: Cartesian categories

Recall that a Cartesian category is a category **C** with a terminal object $\mathbf{1} \in \mathbf{C}$ and binary products $X \times Y \in \mathbf{C}$ for all objects $X \times Y$.

(So that **C** has all finite products)

- Given $X, Y \in \mathbf{C}$, we write:
  - $\pi_{X,Y} \ \in \ \mathbf{C}(X \times Y, X)$ (1st projection)
  - $\pi'_{X,Y} \ \in \ \mathbf{C}(X \times Y, Y)$ (2nd projection)
  - $\tau_{X,Y} \ := \ \langle \pi'_{X,Y}, \pi_{X,Y} \rangle \ \in \ \mathbf{C}(X \times Y, Y \times X)$ (iso of exchange)
  - $\delta_X \ := \ \langle \mathrm{id}_X, \mathrm{id}_X \rangle \ \in \ \mathbf{C}(X, X \times X)$ (arrow of duplication)

First-order hyperdoctrines: definition

Let $\mathbf{C}$ be a Cartesian category

Definition (First-order hyperdoctrine)

A first-order hyperdoctrine over $\mathbf{C}$ is a functor $\mathrm{P} : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ such that:

(1) For all $Z, X \in \mathbf{C}$, the map $\mathrm{P}(\pi_{Z,X}) : \mathrm{P}(Z) \to \mathrm{P}(Z \times X)$ has left and right adjoints $(\exists X)_{|Z}, (\forall X)_{|Z} : \mathrm{P}(Z \times X) \to \mathrm{P}(Z)$

(2) The following diagrams (Beck-Chevalley conditions)

$$\begin{array}{ccc} \mathrm{P}(Z \times X) & \xrightarrow{(\exists X)_{|Z}} & \mathrm{P}(Z) \\ {\scriptstyle \mathrm{P}(f\times \mathrm{id}_X)}\uparrow & & \uparrow{\scriptstyle \mathrm{P}(f)} \\ \mathrm{P}(Z' \times X) & \xrightarrow[(\exists X)_{|Z'}]{} & \mathrm{P}(Z') \end{array} \qquad \begin{array}{ccc} \mathrm{P}(Z \times X) & \xrightarrow{(\forall X)_{|Z}} & \mathrm{P}(Z) \\ {\scriptstyle \mathrm{P}(f\times \mathrm{id}_X)}\uparrow & & \uparrow{\scriptstyle \mathrm{P}(f)} \\ \mathrm{P}(Z' \times X) & \xrightarrow[(\forall X)_{|Z'}]{} & \mathrm{P}(Z') \end{array}$$

commute for all $X, Z, Z' \in \mathbf{C}$ and $f \in \mathbf{C}(Z, Z')$

(3) Each $X \in \mathbf{C}$ has an equality predicate $(=_X) \in \mathrm{P}(X \times X)$, such that:

$$(=_X) \leq q \quad \Leftrightarrow \quad \top \leq \mathrm{P}(\delta_X)(q) \qquad (q \in \mathrm{P}(X \times X))$$

First-order hyperdoctrines: some intuitions (1/5)

Intuitively, a first-order hyperdoctrine $\mathrm{P} : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ is an abstract description of a particular intuitionistic or classical theory. Note that such a description also applies to models, that can be viewed as theories.

In this framework:

- The Cartesian category $\mathbf{C}$ represents the domain of the discourse
  - The objects of $\mathbf{C}$ represent types, or contexts
  - The arrows of $\mathbf{C}$ represent functions, or substitutions
  - The Cartesian product $X \times Y$ (in $\mathbf{C}$) represents the product of two types, or the concatenation of two contexts
  - The terminal object $\mathbf{1}$ ($\in \mathbf{C}$) represents the singleton type, or the empty context
- The (contravariant) functor $\mathrm{P} : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ associates to each object $X \in \mathbf{C}$ the Heyting algebra $\mathrm{P}(X)$ of predicates over X

First-order hyperdoctrines: some intuitions (2/5)

- The (contravariant) functor $\mathrm{P} : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ associates to each object $X \in \mathbf{C}$ the Heyting algebra $\mathrm{P}(X)$ of predicates over X

  - Each predicate $p \in \mathrm{P}(X)$ can be viewed as an abstract formula $p(x)$ depending on a variable $x : X$. Intuitively:

$$p \leq q \quad \text{means:} \quad (\forall x : X)\,(p(x) \Rightarrow q(x))$$
$$p = q \quad \text{means:} \quad (\forall x : X)\,(p(x) \Leftrightarrow q(x))$$

    (So that in this description, the ordering $\leq$ represents inclusion whereas equality represent extensional equality of predicates)

  - $\mathrm{P}(X)$ is a Heyting algebra, which means that predicates $p, q \in \mathrm{P}(X)$ can be assembled using the constructions

$$\bot, \quad \top, \quad p \wedge q, \quad p \vee q, \quad p \to q$$

    The axioms of Heyting algebras express that all the deduction rules of intuitionistic propositional calculus are valid

First-order hyperdoctrines: some intuitions (3/5)

- The correspondence $X \mapsto \mathrm{P}(X)$ is functorial, since each arrow $f \in \mathbf{C}(X, Y)$ induces a substitution map $\mathrm{P}(f) : \mathrm{P}(Y) \to \mathrm{P}(X)$:
  - Given $p \in \mathrm{P}(Y)$, the predicate $\mathrm{P}(f)(p) \in \mathrm{P}(X)$ represents the pre-image of p by f: $\mathrm{P}(f)(p) \equiv$ "$p \circ f$"
  - Or, if we see p as a formula $p(y)$ (in the context $y : Y$) then $\mathrm{P}(f)(p)$ is the formula $p(y)\{y := f(x)\}$ (in the context $x : X$)
- The fact that $\mathrm{P}(f) : \mathrm{P}(Y) \to \mathrm{P}(X)$ is a morphism of HAs expresses that substitution commutes with all connectives:

$$\begin{aligned}
(p(y) \wedge q(y))\{y := f(x)\} &\equiv p(f(x)) \wedge q(f(x)) \\
(p(y) \vee q(y))\{y := f(x)\} &\equiv p(f(x)) \vee q(f(x)) \\
(p(y) \to q(y))\{y := f(x)\} &\equiv p(f(x)) \to q(f(x))
\end{aligned}$$

- Identities $\mathrm{P}(\mathrm{id}_X) = \mathrm{id}_{\mathrm{P}(X)}$ and $\mathrm{P}(g \circ f) = \mathrm{P}(f) \circ \mathrm{P}(g)$ express that the operation of substitution (or pre-image) is contravariant

First-order hyperdoctrines: some intuitions (4/5)

- Axiom (1) says that the map $\mathrm{P}(\pi_{Z,X}) : \mathrm{P}(Z) \to \mathrm{P}(Z \times X)$ associated to the 1st projection $\pi_{Z,X} \in \mathbf{C}(Z \times X, Z)$ has both adjoints $(\exists X)_{|Z}, (\forall X)_{|Z} : \mathrm{P}(Z \times X) \to \mathrm{P}(Z)$
  - Recall that these adjoints are unique and monotonic; but in general, they are *not* morphisms of Heyting algebras.
  - Given $p \in \mathrm{P}(Z \times X)$:

$$\begin{array}{lll} (\exists X)_{|Z}(p) & \text{means:} & (\exists x : X)\, p(z,x) \\ (\forall X)_{|Z}(p) & \text{means:} & (\forall x : X)\, p(z,x) \end{array} \qquad \text{(in context } z : Z\text{)}$$

  - Given $p \in \mathrm{P}(Z \times X)$ and $q \in \mathrm{P}(Z)$, the adjunctions

$$\begin{array}{lcl} (\exists X)_{|Z}(p) \le q & \Leftrightarrow & p \le \mathrm{P}(\pi_{Z,X})(q) \\ q \le (\forall X)_{|Z}(p) & \Leftrightarrow & \mathrm{P}(\pi_{Z,X})(q) \le p \end{array}$$

    represent the logical equivalences:

$$\begin{array}{lcl} (\forall z : Z)[(\exists x : X)\, p(z,x) \Rightarrow q(z)] & \Leftrightarrow & (\forall z : Z,\ x : X)[p(z,x) \Rightarrow q(z)] \\ (\forall z : Z)[q(z) \Rightarrow (\forall x : X)\, p(z,x)] & \Leftrightarrow & (\forall z : Z,\ x : X)[q(z) \Rightarrow p(z,x)] \end{array}$$

First-order hyperdoctrines: some intuitions (5/5)

- The Beck-Chevalley conditions (2)

$$\begin{array}{ccc} P(Z\times X) & \xrightarrow{(\exists X)_{|Z}} & P(Z) \\ \uparrow{\scriptstyle P(f\times \mathrm{id}_X)} & & \uparrow{\scriptstyle P(f)} \\ P(Z'\times X) & \xrightarrow[(\exists X)_{|Z'}]{} & P(Z') \end{array} \qquad \begin{array}{ccc} P(Z\times X) & \xrightarrow{(\forall X)_{|Z}} & P(Z) \\ \uparrow{\scriptstyle P(f\times \mathrm{id}_X)} & & \uparrow{\scriptstyle P(f)} \\ P(Z'\times X) & \xrightarrow[(\forall X)_{|Z'}]{} & P(Z') \end{array}$$

express the behavior of substitution w.r.t. quantifiers:

$$\begin{array}{rcl} ((\exists x : X)\, p(z',x))\{z' := f(z)\} & \equiv & (\exists x : X)\big(p(z',x)\{z' := f(z), x := x\}\big) \\ ((\forall x : X)\, p(z',x))\{z' := f(z)\} & \equiv & (\forall x : X)\big(p(z',x)\{z' := f(z), x := x\}\big) \end{array}$$

- Axiom (3) expresses that the map $P(\delta_X) : P(X\times X) \to P(X)$ has a left adjoint $(=_X) \in P(X\times X)$ at the point $\top \in P(X)$:

$$(=_X) \ \le\ q \quad\Leftrightarrow\quad \top \ \le\ P(\delta_X)(q) \qquad\qquad (q \in P(X\times X))$$

The above adjunction corresponds to the logical equivalence:

$$(\forall x, y : X)[x = y \Rightarrow q(x,y)] \quad\Leftrightarrow\quad (\forall x : X)[\top \Rightarrow q(x,x)]$$

First-order hyperdoctrines: definition (recall)

Let **C** be a Cartesian category

Definition (First-order hyperdoctrine)

A first-order hyperdoctrine over **C** is a functor $\mathrm{P} : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ such that:

(1) For all $Z, X \in \mathbf{C}$, the map $\mathrm{P}(\pi_{Z,X}) : \mathrm{P}(Z) \to \mathrm{P}(Z \times X)$ has left and right adjoints $(\exists X)_{|Z}, (\forall X)_{|Z} : \mathrm{P}(Z \times X) \to \mathrm{P}(Z)$

(2) The following diagrams (Beck-Chevalley conditions)

$$\begin{array}{ccc} \mathrm{P}(Z \times X) & \xrightarrow{(\exists X)_{|Z}} & \mathrm{P}(Z) \\ \uparrow{\scriptstyle \mathrm{P}(f \times \mathrm{id}_X)} & & \uparrow{\scriptstyle \mathrm{P}(f)} \\ \mathrm{P}(Z' \times X) & \xrightarrow[(\exists X)_{|Z'}]{} & \mathrm{P}(Z') \end{array} \qquad \begin{array}{ccc} \mathrm{P}(Z \times X) & \xrightarrow{(\forall X)_{|Z}} & \mathrm{P}(Z) \\ \uparrow{\scriptstyle \mathrm{P}(f \times \mathrm{id}_X)} & & \uparrow{\scriptstyle \mathrm{P}(f)} \\ \mathrm{P}(Z' \times X) & \xrightarrow[(\forall X)_{|Z'}]{} & \mathrm{P}(Z') \end{array}$$

commute for all $X, Z, Z' \in \mathbf{C}$ and $f \in \mathbf{C}(Z, Z')$

(3) Each $X \in \mathbf{C}$ has an equality predicate $(=_X) \in \mathrm{P}(X \times X)$, such that:

$$(=_X) \leq q \quad \Leftrightarrow \quad \top \leq \mathrm{P}(\delta_X)(q) \qquad (q \in \mathrm{P}(X \times X))$$

First-order hyperdoctrines: some properties (1/2)

Let $\mathrm{P} : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ be a first-order hyperdoctrine

- Using equality predicates $(=_X) \in \mathrm{P}(X \times X)$, one can show more generally that all substitution maps $\mathrm{P}(f) : \mathrm{P}(Y) \to \mathrm{P}(X)$ have left and right adjoints $\exists(f), \forall(f) \ : \ \mathrm{P}(X) \to \mathrm{P}(Y)$

- Intuitively, given a predicate $p \in \mathrm{P}(X)$, the two predicates $\exists(f)(p), \forall(f)(p) \in \mathrm{P}(Y)$ are defined by:

$$\begin{aligned} \big(\exists(f)(p)\big)(y) &\equiv (\exists x : X)\,(y = f(x) \wedge p(x)) \\ \big(\forall(f)(p)\big)(y) &\equiv (\forall x : X)\,(y = f(x) \Rightarrow p(x)) \end{aligned}$$

First-order hyperdoctrines: some properties (2/2)

- In the definition of hyperdoctrines, some authors require that the Beck-Chevalley condition holds for all pullback squares in **C**:

$$\begin{array}{ccc} X & \xrightarrow{f_1} & X_1 \\ {\scriptstyle f_2}\downarrow & \lrcorner & \downarrow{\scriptstyle g_1} \\ X_2 & \xrightarrow[g_2]{} & Y \end{array} \quad\Rightarrow\quad \begin{array}{ccc} \mathrm{P}(X) & \xrightarrow{\exists(f_1)} & \mathrm{P}(X_1) \\ {\scriptstyle \mathrm{P}(f_2)}\uparrow & & \uparrow{\scriptstyle \mathrm{P}(g_1)} \\ \mathrm{P}(X_2) & \xrightarrow[\exists(g_2)]{} & \mathrm{P}(Y) \end{array} \qquad \begin{array}{ccc} \mathrm{P}(X) & \xrightarrow{\forall(f_1)} & \mathrm{P}(X_1) \\ {\scriptstyle \mathrm{P}(f_2)}\uparrow & & \uparrow{\scriptstyle \mathrm{P}(g_1)} \\ \mathrm{P}(X_2) & \xrightarrow[\forall(g_2)]{} & \mathrm{P}(Y) \end{array}$$

  (full Beck-Chevalley condition)

- **Beware!** The full Beck-Chevalley is strictly stronger than the Beck-Chevalley condition restricted to the projections
  (there are counter-examples with some syntactic hyperdoctrines)

- However, this stronger condition holds in most models, and in particular in all forcing/realizability/implicative triposes

Triposes: definition

Let **C** be a Cartesian closed category

Definition (Tripos)

A tripos over **C** is a first-order hyperdoctrine $\mathrm{P} : \mathbf{C}^{\mathrm{op}} \to \mathbf{HA}$ given with an object $\mathrm{Prop} \in \mathbf{C}$ and a generic predicate $\mathrm{tr} \in \mathrm{P}(\mathrm{Prop})$, such that:

For all $X \in \mathbf{C}$, each predicate $p \in \mathrm{P}(X)$ is represented by an arrow $f_p \in \mathbf{C}(X, \mathrm{Prop})$ (not necessarily unique) such that:

$$P(\mathrm{tr})(f_p) \;=\; p$$

Intuitively:

- The Cartesian closed category **C** is a model of the simply-typed λ-calculus
- Object $\mathrm{Prop} \in \mathbf{C}$ is the type of propositions
- Generic predicate $\mathrm{tr} \in \mathrm{P}(\mathrm{Prop})$ is the truth predicate
- For each predicate $p \in \mathrm{P}(X)$, the corresponding arrow $f_p \in \mathbf{C}(X, \mathrm{Prop})$ is a propositional function representing p: $\;\mathrm{tr}(f_p(x)) \;\equiv\; p(x)$
- In what follows, we shall only consider triposes over the c.c.c. **Set**

Example: forcing triposes

Proposition and definition (Forcing triposes)

Given a complete Heyting (or Boolean) algebra H:

1. The functor $\mathrm{P} := H^{(-)} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ is a tripos
2. For all $X, Y \in \mathbf{Set}$, $f : X \to Y$:
   - $\mathrm{P}(X) := H^X$ is a complete HA
   - $\mathrm{P}(f) : \mathrm{P}(Y) \to \mathrm{P}(X)$ is a morphism of complete HAs
3. $\mathrm{Prop} := H$ and $\mathrm{tr} := \mathrm{id}_H$ (generic predicate)

Such a tripos is called a forcing tripos

- Forcing triposes are the ones underlying Kripke (or Cohen) forcing

Product of a family of implicative structures (1/2)

Given a family of implicative structures $(\mathscr{A}_i)_{i\in I} = (\mathscr{A}_i, \preccurlyeq_i, \to_i)_{i\in I}$

- The product $\mathscr{A} = \prod_{i\in I} \mathscr{A}_i$ of the family $(\mathscr{A}_i)_{i\in I} = (\mathscr{A}_i, \preccurlyeq_i, \to_i)_{i\in I}$ is clearly an implicative structure, where:

$$(a_i)_{i\in I} \preccurlyeq (b_i)_{i\in I} \quad\equiv\quad (\forall i\in I)\, a_i \preccurlyeq_i b_i \qquad \text{(product ordering)}$$

$$(a_i)_{i\in I} \to (b_i)_{i\in I} \quad:=\quad (a_i \to_i b_i)_{i\in I} \qquad \text{(componentwise)}$$

Proposition (Properties of the product $\prod_{i\in I} \mathscr{A}_i$)

In the product $\mathscr{A} = \prod_{i\in I} \mathscr{A}_i$, we have:

1. $ab \;=\; (a_i b_i)_{i\in I}$ for all $a, b \in \mathscr{A}$
2. $(t)^{\mathscr{A}} \;=\; \left((t)^{\mathscr{A}_i}\right)_{i\in I}$ for all closed λ-terms t
3. $\mathbf{S}^{\mathscr{A}} = \left(\mathbf{S}^{\mathscr{A}_i}\right)_{i\in I}$ $\quad\mathbf{K}^{\mathscr{A}} = \left(\mathbf{K}^{\mathscr{A}_i}\right)_{i\in I}$ $\quad ⅽⅽ^{\mathscr{A}} = \left(ⅽⅽ^{\mathscr{A}_i}\right)_{i\in I}$ etc.
4. $a \times b = (a_i \times b_i)_{i\in I}$, $\quad a + b = (a_i + b_i)_{i\in I}$ for all $a, b \in \mathscr{A}$

Product of a family of implicative structures (2/2)

Given a family of implicative structures $(\mathscr{A}_i)_{i\in I} = (\mathscr{A}_i, \preccurlyeq_i, \to_i)_{i\in I}$

- The product $S = \prod_{i\in I} S_i$ of a family of separators $(S_i \subseteq \mathscr{A}_i)_{i\in I}$ is clearly a separator of the product $\mathscr{A} = \prod_{i\in I} \mathscr{A}_i$
- Moreover, we have: $a \vdash_S b \;\Leftrightarrow\; (\forall i \in I)\, a_i \vdash_{S_i} b_i$ (for all $a, b \in \mathscr{A}$)

Proposition (Factorization of the quotient)

$$\mathscr{A}/S \;=\; \Big(\prod_{i\in I} \mathscr{A}_i\Big) \Big/ \Big(\prod_{i\in I} S_i\Big) \;\cong\; \prod_{i\in I} (\mathscr{A}_i/S_i) \qquad \text{(iso. in \textbf{HA})}$$

- **Beware!** We only have the inclusions

$$S^0(\mathscr{A}) \;\subseteq\; \prod_{i\in I} S^0(\mathscr{A}_i) \qquad \text{(intuitionistic core)}$$

$$S^0_K(\mathscr{A}) \;\subseteq\; \prod_{i\in I} S^0_K(\mathscr{A}_i) \qquad \text{(classical core)}$$

Power of an implicative structure

Given an implicative structure $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ and a set I, we write

$$\mathscr{A}^I \;:=\; (\mathscr{A}^I, \preccurlyeq^I, \to^I) \;:=\; \prod_{i\in I} \mathscr{A} \qquad \text{(power implicative structure)}$$

Each separator $S \subseteq \mathscr{A}$ induces two separators in $\mathscr{A}^I$:

- The power separator $S^I \;:=\; \prod_{i\in I} S \;\subseteq\; \mathscr{A}^I$,

  for which we have: $\mathscr{A}^I/S^I \;\cong\; (\mathscr{A}/S)^I$

- The uniform power separator $S[I] \;\subseteq\; S^I \;\subseteq\; \mathscr{A}^I$ defined by:

  $$S[I] \;:=\; \{(a_i)_{i\in I} \in \mathscr{A}^I \;:\; (\exists s \in S)(\forall i \in I)\, s \preccurlyeq a_i\} \;=\; \uparrow\delta(S)$$

  where $\uparrow\delta(S)$ is the upwards closure (in $\mathscr{A}^I$) of the image of S through the canonical map $\delta : \mathscr{A} \to \mathscr{A}^I$ defined by $\delta(a) := (i \mapsto a) \in \mathscr{A}^I$ for all $a \in \mathscr{A}$

- In general, the inclusion $S[I] \;\subseteq\; S^I$ is strict!

Properties of the uniform power separator

Let $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to)$ be an implicative structure, and I a set.

Each separator $S \subseteq \mathscr{A}$ induces a uniform power separator $S[I] \subseteq \mathscr{A}^I$

Proposition (Entailment w.r.t. $S[I]$)

For all families $a = (a_i)_{i \in I},\ b = (b_i)_{i \in I} \in \mathscr{A}^I$, we have:

$$a \vdash_{S[I]} b \quad \Leftrightarrow \quad (a \to b) \in S[I] \quad \Leftrightarrow \quad \bigwedge_{i \in I} (a_i \to b_i) \in S$$

$$a \dashv\vdash_{S[I]} b \quad \Leftrightarrow \quad (a \leftrightarrow b) \in S[I] \quad \Leftrightarrow \quad \bigwedge_{i \in I} (a_i \leftrightarrow b_i) \in S$$

Recall that $a \leftrightarrow b \ := \ (a \to b) \times (b \to a)$ (in any implicative structure)

We can also notice that:

- $S^0(\mathscr{A}^I) \ = \ S^0(\mathscr{A})[I] \ \subseteq \ (S^0(\mathscr{A}))^I$ (intuitionistic core of $\mathscr{A}^I$)
- $S^0_K(\mathscr{A}^I) \ = \ S^0_K(\mathscr{A})[I] \ \subseteq \ (S^0_K(\mathscr{A}))^I$ (classical core of $\mathscr{A}^I$)

Tripos associated to an implicative algebra (1/2)

Let $(\mathscr{A}, S) = (\mathscr{A}, \preccurlyeq, \to, S)$ be an implicative algebra

For each set I, we let $\mathrm{P}(I) \ := \ \mathscr{A}^I / S[I]$

- The poset $(\mathrm{P}(I), \leq_{S[I]})$ is a Heyting algebra, where:

$$
\begin{array}{rcl}
[a] \to [b] & = & [(a_i \to b_i)_{i \in I}] \\
[a] \wedge [b] & = & [(a_i \times b_i)_{i \in I}] \qquad \top \ = \ [(\top)_{i \in I}] \\
[a] \vee [b] & = & [(a_i + b_i)_{i \in I}] \qquad \bot \ = \ [(\bot)_{i \in I}]
\end{array}
$$

- The correspondence $I \mapsto \mathrm{P}(I)$ is functorial:
  - Each $f : I \to J$ induces a substitution map $\mathrm{P}(f) : \mathrm{P}(J) \to \mathrm{P}(I)$:

$$\mathrm{P}(f)([(a_j)_{j \in J}]) \ := \ [(a_{f(i)})_{i \in I}] \ \in \ \mathrm{P}(I)$$

  - The map $\mathrm{P}(f) : \mathrm{P}(J) \to \mathrm{P}(I)$ is a morphism of Heyting algebras
  - $\mathrm{P}(\mathrm{id}_I) = \mathrm{id}_{\mathrm{P}(I)}$ and $\mathrm{P}(g \circ f) = \mathrm{P}(f) \circ \mathrm{P}(g)$ (contravariance)

Therefore: $\mathrm{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ is a (contravariant) functor

Tripos associated to an implicative algebra (2/2)

Theorem (Associated tripos)

The functor $\mathrm{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ is a tripos

Recall: Tripos = categorical model of higher-order logic

- Each substitution map $\mathrm{P}(f) : \mathrm{P}(J) \to \mathrm{P}(I)$ has both left and right adjoints $\exists(f), \forall(f) : \mathrm{P}(I) \to \mathrm{P}(J)$:

$$\exists(f)([(a_i)_{i\in I}]) \;:=\; \left[\left(\exists_{i\in f^{-1}(j)}\, a_i\right)_{j\in J}\right] \;\in\; \mathrm{P}(J)$$

$$\forall(f)([(a_i)_{i\in I}]) \;:=\; \left[\left(\forall_{i\in f^{-1}(j)}\, a_i\right)_{j\in J}\right] \;\in\; \mathrm{P}(J)$$

(+ satisfies the full Beck-Chevalley condition)

- There is a propositional object $\mathrm{Prop} \in \mathbf{Set}$ together with a generic predicate $\mathrm{tr} \in \mathrm{P}(\mathrm{Prop})$:

$$\mathrm{Prop} := \mathscr{A} \qquad\qquad \mathrm{tr} := [\mathrm{id}_{\mathscr{A}}] \in \mathrm{P}(\mathrm{Prop})$$

To sum up...

- The above construction encompasses many well-known tripos constructions:
  - Forcing triposes, which correspond to the case where $(\mathscr{A}, \preccurlyeq, \to)$ is a complete Heyting/Boolean algebra, and $S = \{\top\}$ (i.e. no quotient)
  - Triposes induced by total combinatory algebras... (int. realizability)
    ... and even by partial combinatory algebras, via some completion trick
  - Triposes induced by abstract Krivine structures (class. realizability)
- As for any tripos, each implicative tripos can be turned into a topos via the standard tripos-to-topos construction
- **Question:** What do implicative triposes bring new w.r.t.
  - Forcing triposes (intuitionistic or classical)?
  - Intuitionistic realizability triposes?
  - Classical realizability triposes?

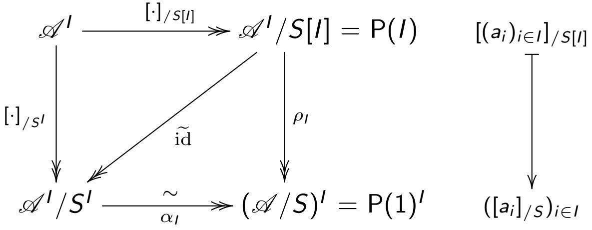
The fundamental diagram (1/3)

Given an implicative algebra $\mathscr{A} = (\mathscr{A}, \preccurlyeq, \to, S)$ and a set I, the separator $S \subseteq \mathscr{A}$ induces two separators in $\mathscr{A}^I$:

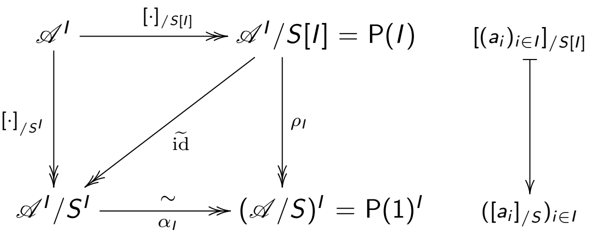
- The power separator $S^I \subseteq \mathscr{A}^I$
- The uniform power separator $S[I] \subseteq S^I \subseteq \mathscr{A}^I$ defined by:

$$S[I] := \big\{(a_i)_{i\in I} \in \mathscr{A}^I \ : \ (\exists s \in S)(\forall i \in I)\, s \preccurlyeq a_i\big\}$$

We thus get the following (commutative) diagram: (in **Set**/**HA**)



The fundamental diagram (2/3)

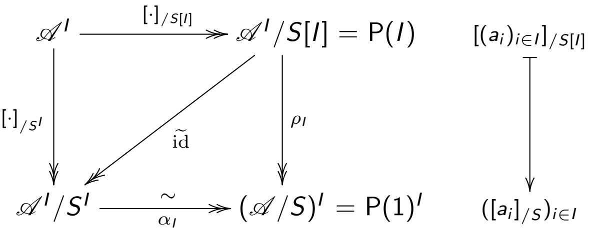


Proposition

The following are equivalent:

1. The map $\rho_I : (\mathscr{A}^I/S[I]) \to (\mathscr{A}/S)^I$ is injective
2. The map $\rho_I : (\mathscr{A}^I/S[I]) \to (\mathscr{A}/S)^I$ is an isomorphism (of HAs)
3. $S[I] = S^I$
4. The separator $S \subseteq \mathscr{A}$ is closed under all I-indexed meets.

The fundamental diagram (3/3)



Proof.

- Recall that in **HA**, a morphism is an iso if and only if it is bijective. Since ρ is surjective and α_I is an iso, it is clear that:

$$(1)\quad \rho \text{ injective} \quad\Leftrightarrow\quad (2)\quad \rho \text{ iso.} \quad\Leftrightarrow\quad \widetilde{\mathrm{id}} \text{ iso.} \quad\Leftrightarrow\quad (3)\quad S[I] = S^I$$

- (3) $\Rightarrow$ (4) Let $(a_i)_{i\in I} \in S^I$. Since $S^I = S[I]$ (by (3)), there is $s \in S$ such that $s \preccurlyeq a_i$ for all $i \in I$. Hence $s \preccurlyeq \curlywedge_{i\in I} a_i \in S$.
- (4) $\Rightarrow$ (3) Let $(a_i)_{i\in I} \in S^I$. By (4), we have that $s := \curlywedge_{i\in I} a_i \in S$. Since $s \preccurlyeq a_i$ for all $i \in I$, we get $(a_i)_{i\in I} \in S[I]$. Therefore: $S^I = S[I]$. □

Forcing triposes (recall)

Proposition and definition (Forcing triposes)

Given a complete Heyting (or Boolean) algebra H:

1. The functor $\mathrm{P} := H^{(-)} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ is a tripos
2. For all $I, J \in \mathbf{Set}$, $f : I \to J$:
   - $\mathrm{P}(I) := H^I$ is a complete HA
   - $\mathrm{P}(f) : \mathrm{P}(J) \to \mathrm{P}(I)$ is a morphism of complete HAs
3. $\mathrm{Prop} := H$ and $\mathrm{tr} := \mathrm{id}_H$ (generic predicate)

Such a tripos is called a forcing tripos

- Forcing triposes are the ones underlying Kripke (or Cohen) forcing
- Each forcing tripos (induced by H) can be seen as an implicative tripos, constructed from the implicative algebra

$$(\mathscr{A}, \preccurlyeq, \to, S) := (H, \leq_H, \to_H, \{\top_H\})$$

Isomorphism of triposes

Definition (Isomorphism of triposes)

Two triposes $\mathrm{P}, \mathrm{P}' : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ are isomorphic when there is a natural isomorphism $\beta : \mathrm{P} \Rightarrow \mathrm{P}'$ (in the category **HA**):

$$\begin{array}{ccccc}
I & \quad & \mathrm{P}(I) & \xrightarrow[\sim]{\beta_I} & \mathrm{P}'(I) \\
\downarrow f & & \uparrow \mathrm{P}(f) & & \uparrow \mathrm{P}'(f) \\
J & & \mathrm{P}(J) & \xrightarrow[\beta_J]{\sim} & \mathrm{P}'(J)
\end{array}$$

- We have seen that each Heyting tripos is isomorphic to a particular implicative tripos, taking $(\mathscr{A}, \preccurlyeq, \to, S) := (H, \leq_H, \to_H, \{\top\})$
- But more generally, what are the implicative triposes that are isomorphic to a forcing tripos?

Characterizing forcing triposes (1/4)

Theorem

Let P : $\mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$ be the tripos induced by an implicative algebra $(\mathscr{A}, \preccurlyeq, \to, S)$. Then the following are equivalent:

1. The tripos P is isomorphic to a forcing tripos
2. The separator $S \subseteq \mathscr{A}$ is a principal filter of $\mathscr{A}$
3. The separator $S \subseteq \mathscr{S}$ is finitely generated and $\pitchfork^{\mathscr{A}} \in S$

Remark: These conditions do not imply that $(\mathscr{A}, \preccurlyeq, \to)$ is a Heyting algebra!
Counter-example: Krivine realizability with an instruction $\pitchfork$ (in the separator)

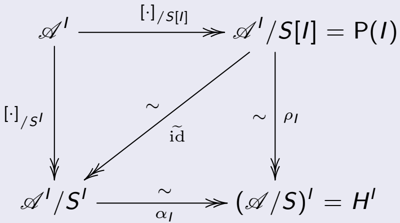
Proof.

- We have already seen that (3) $\Leftrightarrow$ (2).
- So it remains to prove that (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) (...)

Characterizing forcing triposes (2/4)

Proof (continued).

- (2) $\Rightarrow$ (1) When $S \subseteq \mathscr{A}$ is a principal filter of $\mathscr{A}$, we have seen that $H := \mathscr{A}/S$ is a complete Heyting algebra. Moreover, since S is closed under arbitrary meets, the arrow ρ_I of the fundamental diagram



is an isomorphism of (complete) Heyting algebras for all sets I. It is also clearly natural in I, so that we can take $\beta_I := \rho_I$. (...)

Characterizing forcing triposes (3/4)

Proof (continued).

- (1) ⇒ (2) Assume that there is a natural isomorphism $\beta_I : \mathrm{P}(I) \xrightarrow{\sim} H^I$ (in I) for some complete Heyting algebra H. In particular, we have $\beta_1 : \mathrm{P}(1) \xrightarrow{\sim} H^1 = H$, so that $\mathscr{A}/S = \mathrm{P}(1) \cong H$ is a complete HA.

  Now, fix a set I, and write $c_i := \{0 \mapsto i\} : 1 \to I$ for each $i \in I$.

  Via the two (contravariant) functors P, $H^{(-)} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{HA}$, we easily check that the arrow $c_i : 1 \to I$ is mapped to:

  $$\begin{array}{lrcccl} & \mathrm{P}(c_i) & = & \rho_i & : & \mathscr{A}/S[I] \to \mathscr{A}/S \\ \text{and} & H^{(c_i)} & = & \pi_i & : & H^I \to H \end{array}$$

  where:

  - ρ_i is the ith component of the surjection $\rho_I : \mathscr{A}^I/S[I] \twoheadrightarrow (\mathscr{A}/S)^I$ of the fundamental diagram, given by: $\rho_i([a]_{/S[I]}) = [a_i]_{/S}$
  - π_i is the ith projection from H^I to H (...)

Characterizing forcing triposes (4/4)

Proof (continued).

- (1) $\Rightarrow$ (2) (continued) We get the following commutative diagrams:

$$
\begin{array}{c}
1 \\
\big\downarrow{\scriptstyle c_i} \\
I
\end{array}
\qquad
\begin{array}{ccc}
\mathscr{A}/S & \xrightarrow[\sim]{\beta_1} & H \\
{\scriptstyle \mathrm{P}(c_i)=\rho_i}\big\uparrow & & \big\uparrow{\scriptstyle \pi_i = H^{(c_i)}} \\
\mathscr{A}^I/S[I] & \xrightarrow[\beta_I]{\sim} & H^I
\end{array}
\qquad
\begin{array}{ccc}
(\mathscr{A}/S)^I & \xrightarrow[\sim]{\beta_1^I} & H^I \\
{\scriptstyle \rho_I = \langle \rho_i \rangle_{i \in I}}\big\uparrow & & \sim\big\uparrow{\scriptstyle \mathrm{id} = \langle \pi_i \rangle_{i \in I}} \\
\mathscr{A}^I/S[I] & \xrightarrow[\beta_I]{\sim} & H^I
\end{array}
$$

  - 1st commutative square (for $i \in I$) comes from the naturality of β
  - 2nd commutative square is deduced from the first one by glueing the arrows ρ_i and π_i for all indices $i \in I$

From the 2nd commutative square, it is clear that $\rho_I : \mathscr{A}^I/S[I] \to (\mathscr{A}/S)^I$ is an isomorphism for all sets I. Therefore, the separator $S \subseteq \mathscr{A}$ is closed under arbitrary meets, which means that it is a principal filter. □

Plan

Conclusion (1/2)

We introduced implicative algebras, a simple algebraic structure that is common to forcing and realizability (intuitionistic & classical)

- Relies on the fundamental idea that truth values can be manipulated as generalized realizers (via the operations of the λ-calculus)

Proof = Program = Type = Formula

- Criterion of truth given by a separator (generalizing filters)
- Each implicative algebra induces a tripos, thus encompassing:
  - All forcing triposes (intuitionistic & classical)
  - Most intuitionistic realizability triposes
  - All classical realizability triposes
- In this structure: forcing = non deterministic realizability
- Classical implicative structures have the very same expressiveness as Abstract Krivine Structures (with a much lighter machinery)

Conclusion (2/2)

However, implicative algebras can be used directly to construct models of Zermelo-Fraenkel set theory (ZF/IZF)

- Same technique as for constructing Boolean-valued models of ZF (or realizability models of IZF)
- Technically, the construction is not the same in the intuitionistic case (IZF) and the classical case (ZF) (due to reasons of polarity)
- Classical interpretation of dependent choices (DC) using quote
- A particular model with fascinating properties: the model of threads
  [Krivine 12] *Realizability algebras II: new models of ZF + DC*

Open problems & Future work:

- Structure of classical realizability models of set theory?
- What is the equivalent of the generic set?
- New relative consistency results?