Linear Hyperdoctrines and comodules.

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Introduction: In this exposition, the notion of linear hyperdoctrine is revisited through the study of categories of comodules indexed by coalgebras (Paré - Grunenfelder).

Linear Hyperdoctrines

A $\mathcal{C}$-*indexed category* Φ is by definition a pseudo-functor

$$\Phi : \mathcal{C}^{op} \to \mathbf{Cat}$$

.

The category $\mathcal{C}$ is referred as the *base* of the $\mathcal{C}$-*indexed category* Φ and for each $C \in \mathcal{C}$ the category $\Phi(C)$ is called the *fibre* of Φ at C.

Notation: $\Phi(-) = (-)^*$

Therefore it consists of:

- categories $\Phi(C)$ for each $C \in \mathcal{C}$,
- functors $\Phi(f)$ for each morphism $f : J \to I$ of $\mathcal{C}$,
- natural isomorphism $\alpha_{g,f} : \Phi(g)\Phi(f) \Rightarrow \Phi(fg)$ for every morphism $f : J \to I$, $g : K \to J$ in $\mathcal{C}$
- natural isomorphism $\beta : \Phi(id_C) \to id_{\Phi(C)}$ for every $C \in \mathcal{C}$.

These natural isomorphisms need to satisfy some obvious coherence conditions.

if $f : J \to I$, $g : K \to J$ and $h : M \to K$ then

$$\begin{array}{ccc}
\Phi(h)\Phi(g)\Phi(f) & \xrightarrow{1_h \alpha_{g,f}} & \Phi(h)\Phi(fg) \\
{\scriptstyle \alpha_{h,g} 1_f} \downarrow & & \downarrow {\scriptstyle \alpha_{h,fg}} \\
\Phi(gh)\Phi(f) & \xrightarrow[\alpha_{gh,f}]{} & \Phi(fgh)
\end{array}$$

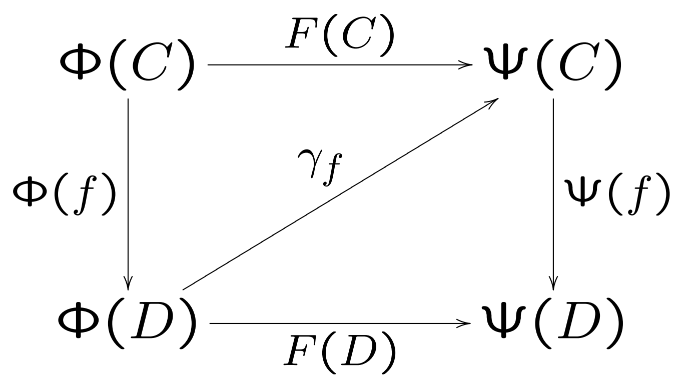
where $\alpha_{g,f} : \Phi(g)\Phi(f) \Rightarrow \Phi(fg)$ is a natural isomorphism.

And if $f : J \to I$ then

$$\alpha_{f,id} = 1_f \beta : \Phi(f)\Phi(id) \to \Phi(f) id_{\Phi(C)}$$

where $\beta : \Phi(id_C) \Rightarrow id_{\Phi(C)}$ is a natural isomorphism.

Definition 1. A $\mathcal{C}$-*indexed functor* $F : \Phi \to \Psi$ of $\mathcal{C}$-indexed categories consists of functors: $F(C) : \Phi(C) \to \Psi(C)$ for every $C \in \mathcal{C}$, such that for each $f : D \to C$, $\Psi(f)F(C) \cong F(D)\Phi(f)$ i.e., there is a natural isomorphism $\gamma_f : \Psi(f)F(C) \Rightarrow F(D)\Phi(f)$ for each f.



subject to some coherence condition.

Also there is the notion of *indexed natural transformation.*

Two basic examples. Given a category $\mathcal{C}$:

- $\Phi : \mathbf{Set}^{op} \to \mathbf{Cat}$, $\Phi(I) = \mathcal{C}^I$ for $\alpha : J \to I$ define $\Phi(\alpha)$ as follows: if $\{A_i\}_{i \in I} \in \mathcal{C}^I$ then $\Phi(\alpha)(\{A_i\}_{i\in I}) = \{A_{\alpha(j)}\}_{j \in J}$

- a functor $F : \mathcal{C} \to \mathcal{D}$ between categories define an indexed functor: $F(I) : \Phi(I) \to \Psi(I)$ by $F(I)(\{A_i\}_{i\in I}) = \{F(A_i)\}_{i\in I}$.

Given a category $\mathcal{C}$:

- $\Phi(I) = \mathcal{C}/I$ and $\Phi(\alpha) : \mathcal{C}/I \to \mathcal{C}/J$ is given by the pullback:

$$\begin{array}{ccc} P & \longrightarrow & A \\ {\scriptstyle \Phi(\alpha)(a)}\Big\downarrow & & \Big\downarrow{\scriptstyle a} \\ J & \xrightarrow[\alpha]{} & I \end{array}$$

Definition 2. A *linear hyperdoctrine* is specified by the following data:

- a category $\mathcal{B}$ with binary product and terminal object (also a C.C.C.) where there is an object U which generates all other objects by finite products, i.e., for every object $B \in \mathcal{B}$ there is a $n \in \mathbb{N}$ with $B = U^n$ (object=Types, morphism=terms)

- A $\mathcal{B}$-indexed category, $\Phi : \mathcal{B}^{op} \to \mathcal{L}$, where $\mathcal{L}$ is the category of intuitionistic linear categories. (object $\phi \in \Phi(A)$=attributes of type A, morphisms $f \in \Phi(A)$= deductions).

- For each object $I \in \mathcal{B}$ we have functors $\exists_I, \forall_I : \Phi(I \times U) \to \Phi(I)$ which are left, right adjoint to the functor $\Phi(\pi_I) : \Phi(I) \to \Phi(I \times U)$, i.e., $\exists_I \dashv \Phi(\pi_I) \dashv \forall_I$. Moreover, given any morphism $f : J \to I$ in $\mathcal{B}$ the following diagram

$$\begin{array}{ccc} \Phi(I \times U) & \xrightarrow{\forall_I} & \Phi(I) \\ {\scriptstyle \Phi(f \times 1_U)} \downarrow & & \downarrow {\scriptstyle \Phi(f)} \\ \Phi(J \times U) & \xrightarrow[\forall_J]{} & \Phi(J) \end{array}$$

conmutes. This last requirement is called *Beck-Chevalley condition*.

Linear Categories

Definition 3. A *monoidal* category, also often called *tensor* category, is a category $\mathcal{V}$ with an identity object $I \in \mathcal{V}$ together with a bifunctor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ and natural isomorphisms $\rho : A \otimes I \overset{\cong}{\to} A$, $\lambda : I \otimes A \overset{\cong}{\to} A$, $\alpha : A \otimes (B \otimes C) \overset{\cong}{\to} (A \otimes B) \otimes C$, satisfying the following coherence commutativity axioms:

$$\begin{CD} A \otimes (I \otimes B) @>{\alpha}>> (A \otimes I) \otimes B \\ @V{1 \otimes \lambda}VV @VV{\rho \otimes 1}V \\ A \otimes B @= A \otimes B \end{CD}$$

and

$$\begin{CD} A \otimes (B \otimes (C \otimes D)) @>{\alpha}>> (A \otimes B) \otimes (C \otimes D) @>{\alpha}>> ((A \otimes B) \otimes C) \otimes D \\ @V{\alpha}VV @. @VV{\alpha}V \\ (A \otimes ((B \otimes C) \otimes D) @>>{\alpha}> @. (A \otimes (B \otimes C)) \otimes D \end{CD}$$

Definition 4. A *symmetric* monoidal category consists of a monoidal category $(\mathcal{V}, \otimes, I, \alpha, \rho, \lambda)$ with a choosen natural isomorphism σ : $A \otimes B \overset{\cong}{\to} B \otimes A$, called symmetry, which satisfies the following coherence axioms:

$$\begin{array}{ccc} A \otimes B & \xrightarrow{\sigma} & B \otimes A \\ & {\scriptstyle id} \searrow \quad \swarrow {\scriptstyle \sigma} & \\ & A \otimes B & \end{array} \qquad \begin{array}{ccc} A \otimes I & \xrightarrow{\sigma} & I \otimes A \\ & {\scriptstyle \rho} \searrow & \downarrow {\scriptstyle \lambda} \\ & & A \end{array}$$

and

$$\begin{array}{ccccc} A \otimes (B \otimes C) & \xrightarrow{\alpha} & (A \otimes B) \otimes C & \xrightarrow{\sigma} & C \otimes (A \otimes B) \\ \downarrow {\scriptstyle 1 \otimes \sigma} & & & & \downarrow {\scriptstyle \alpha} \\ A \otimes (C \otimes B) & \xrightarrow{\alpha} & (A \otimes C) \otimes B & \xrightarrow{\sigma \otimes 1} & (C \otimes A) \otimes B \end{array}$$

commute.

Definition 5. A *closed* monoidal category is a monoidal category $\mathcal{V}$ for which each functor $- \otimes B : \mathcal{V} \to \mathcal{V}$ has a right adjoint $[B, -] : \mathcal{V} \to \mathcal{V}$:

$$\mathcal{V}(A \otimes B, C) \cong \mathcal{V}(A, [B, C])$$

.

Definition 6. A *monoidal functor* $(F, m_{A,B}, m_I)$ between monoidal categories $(\mathcal{V}, \otimes, I, \alpha, \rho, \lambda)$ and $(\mathcal{W}, \otimes', I', \alpha', \rho', \lambda')$ is a functor $F : \mathcal{V} \to \mathcal{W}$ equipped with:

- morphisms $m_{A,B} : F(A) \otimes' F(B) \to F(A \otimes B)$ natural in A and B ,

- for the units morphism $m_I : I' \to F(I)$

which satisfy the following coherence axioms:

$$\begin{array}{ccccc}
FA \otimes' (FB \otimes' FC) & \xrightarrow{1 \otimes' m} & FA \otimes' F(B \otimes C) & \xrightarrow{m} & F(A \otimes (B \otimes C)) \\
\downarrow{\scriptstyle \alpha'} & & & & \downarrow{\scriptstyle F\alpha} \\
(FA \otimes' FB) \otimes FC & \xrightarrow{m \otimes' 1} & F(A \otimes B) \otimes' FC & \xrightarrow{m} & F((A \otimes B) \otimes C)
\end{array}$$

$$\begin{array}{ccc}
FA \otimes' I' & \xrightarrow{\rho'} & FA \\
\downarrow{\scriptstyle 1 \otimes' m} & & \uparrow{\scriptstyle F\rho} \\
FA \otimes' FI & \xrightarrow[m]{} & F(A \otimes I)
\end{array}
\qquad\qquad
\begin{array}{ccc}
I' \otimes' FA & \xrightarrow{\lambda'} & FA \\
\downarrow{\scriptstyle m \otimes' 1} & & \uparrow{\scriptstyle F(\lambda)} \\
FI \otimes' FA & \xrightarrow[m]{} & F(I \otimes A)
\end{array}$$

A monoidal functor is *strong* when m_I and for every A and B $m_{A,B}$ are isomorphisms. It is said to be *strict* when all the $m_{A,B}$ and m_I are identities.

Definition 7. If $\mathcal{V}$ and $\mathcal{W}$ are symmetric monoidal categories with natural maps σ and σ', a *symmetric monoidal functor* is a monoidal functor $(F, m_{A,B}, m_I)$ such that satisfies the following axiom:

$$\begin{array}{ccc} FA \otimes' FB & \xrightarrow{\sigma'} & FB \otimes' FA \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ F(A \otimes B) & \xrightarrow[F(\sigma)]{} & F(B \otimes A) \end{array}$$

Definition 8. A *monoidal natural transformation* $\theta : (F, m) \to (G, n)$ between monoidal functors is a natural transformation $\theta_A : FA \to GA$ such that the following axioms hold:

$$
\begin{array}{ccc}
FA \otimes' FB & \xrightarrow{\quad m \quad} & F(A \otimes B) \\
{\scriptstyle \theta_A \otimes' \theta_B} \downarrow & & \downarrow {\scriptstyle \theta_{A \otimes B}} \\
GA \otimes' GB & \xrightarrow[\quad n \quad]{} & G(A \otimes B)
\end{array}
\qquad\qquad
\begin{array}{ccc}
I' & \xrightarrow{m_I} & FI \\
 & {\scriptstyle n_I} \searrow & \downarrow {\scriptstyle \theta_I} \\
 & & GI
\end{array}
$$

Definition 9. A *monoidal adjunction*

$$(\mathcal{V}, \otimes, I) \underset{(G,n)}{\overset{(F,m)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} (\mathcal{W}, \otimes', I')$$

between two monoidal functors (F, m) and (G, n) consists of an adjunction $(F, G, \eta, \varepsilon)$ in which the unit $\eta : Id \Rightarrow G \circ F$ and the counit $\varepsilon : F \circ G \Rightarrow Id$ are monoidal natural tranformations.

Proposition 1 (Kelly)**.** Let $(F, m) : \mathcal{C} \to \mathcal{C}'$ be a monoidal functor. Then F has a right adjoint G for which the adjunction $(F, m) \dashv (G, n)$ is monoidal if and only if F has a right adjoint $F \dashv G$ and F is strong monoidal.

Since we have that $\mathcal{C}'(FA, B) \cong \mathcal{C}(A, GB)$ then there is a unique $n_{A,B}$ and n_I such that:

$$
\begin{array}{ccc}
F(GA \otimes GB) & \xrightarrow{F(n_{A,B})} & FG(A \otimes' B) \\
\downarrow {\scriptstyle m^{-1}_{GA,GB}} & & \downarrow {\scriptstyle \epsilon_{A \otimes B}} \\
FGA \otimes' FGB & \xrightarrow[\epsilon_A \otimes \epsilon_B]{} & A \otimes' B
\end{array}
\qquad
\begin{array}{ccc}
FI & \xrightarrow{F(n_I)} & FGI' \\
 & {\scriptstyle m^{-1}_I} \searrow & \downarrow {\scriptstyle \epsilon_{I'}} \\
 & & I'
\end{array}
$$

Then using the adjunction we check that this candidates satisfy the definition.

Definition 10 (Benton)**.** A *linear-non-linear category* consists of:

(1) a symmetric monoidal closed category $(\mathcal{C}, \otimes, I, \multimap)$

(2) a category $(\mathcal{B}, \times, 1)$ with finite product

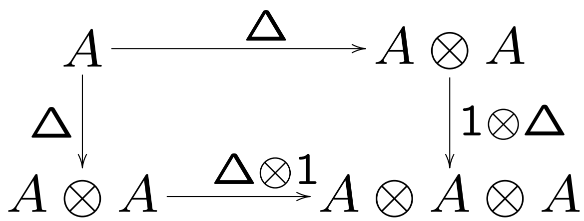
(3) a symmetric monoidal adjunction:

$$(\mathcal{B}, \times, 1) \underset{(G,n)}{\overset{(F,m)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} (\mathcal{C}, \otimes, I)\,.$$

Proposition 2. Every linear-non-linear category gives rise to a linear category. Every linear category defines a linear-non-linear category, where $(\mathcal{B}, \times, 1)$ is the category of coalgebras of the comonad $(!, \varepsilon, \delta)$.

Coalgebras and Comodules

Definition 11. A *coalgebra* C *over a field* $\mathbb{K}$ is a vector space C over a field $\mathbb{K}$ together with $\mathbb{K}$-linear maps $\boldsymbol{\Delta} : C \to C \otimes C$ and $\epsilon : C \to \mathbb{K}$ satisfying the following axioms:



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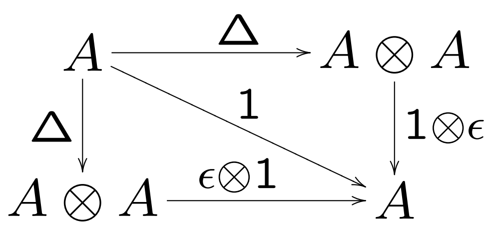
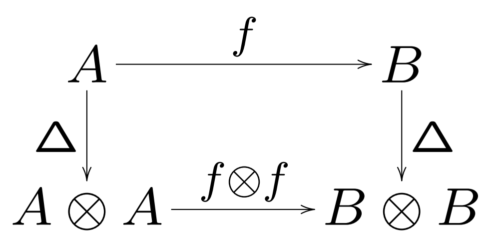
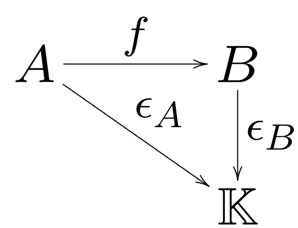


Let $(A, \Delta_A, \epsilon_A)$ and $(B, \Delta_B, \epsilon_B)$ be two coalgebras. A $\mathbb{K}$-linear map $f : A \to B$ is a morphism of coalgebras when the following diagrams are commutative:



and



In this talk we consider cocommutative coalgebras:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ & \searrow^{\Delta} & \downarrow \sigma \\ & & C \otimes C \end{array}$$

where $\sigma(a \otimes b) = b \otimes a$ is the twist map. Because we want to consider a category with finite product.

The terminal object is $\mathbb{K}$ and the unique morphism is ε.

The finite product is given by the tensor:

If $(A, \Delta_A, \epsilon_A)$ and $(B, \Delta_B, \epsilon_B)$ are two coalgebras then:

$(A, \Delta_A, \epsilon_A) \times (B, \Delta_B, \epsilon_B) = (A \otimes B, \Delta_{A\otimes B}, \epsilon_{A\otimes B})$

where $\Delta_{A\otimes B} = (1 \otimes \sigma \otimes 1)(\Delta_A \otimes \Delta_B)$ and $\epsilon_{A\otimes B} = \epsilon_A \otimes \epsilon_B$.

Projection maps:

$\pi_1 : (A, \Delta_A, \epsilon_A) \times (B, \Delta_B, \epsilon_B) \to (A, \Delta_A, \epsilon_A)$

given by:

$\pi_1 = 1 \otimes \epsilon_B$

$\pi_2 : (A, \Delta_A, \epsilon_A) \times (B, \Delta_B, \epsilon_B) \to (A, \Delta_B, \epsilon_B)$

given by:

$\pi_1 = \epsilon_A \otimes 1$

and mediating arrow:

$< f, g >= (f \otimes g)\Delta_C$ if $f : C \to D$ and $f : C \to E$.

Also:

$A \otimes - \dashv \mathsf{Hom}(A, -)$.

i.e.,**CoCoalg** is a cartesian closed category.

Let (D, Δ, ϵ) be a coalgebra. A subspace $S \subseteq D$ is a subcoalgebra when $\Delta(S) \subseteq S \otimes S$.

If $\{S_i\}_{i \in I}$ is a family of subcoalgebras of C then $\sum_{i \in I} S_i$ is a subcoalgebra.

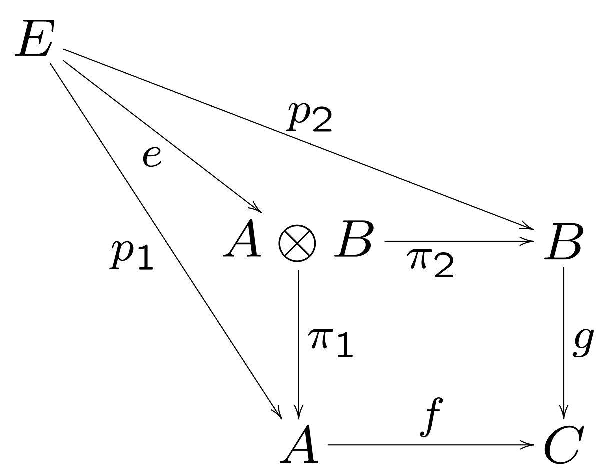
Then **Coalg** has equalizers:

if $f : C \to D$ and $g : C \to D$ we consider the largest subcoalgebra $E \subseteq Ker(f - g)$ i.e.,

$E = \sum_{S \subseteq Ker(f-g)} S$ where S subcoalgebra, and the inclusion map $i : E \to C$.

Therefore we have pull-backs.

If $f : A \to C$ and $g : B \to C$ then:



Definition 12. Let (C, Δ, ϵ) be a coalgebra. A right C-*comodule* M *over a field* $\mathbb{K}$ is a vector space M over a field $\mathbb{K}$ together with $\mathbb{K}$-linear maps $\rho : M \to M \otimes C$ satisfying the following axioms:

$$\begin{array}{ccc} M & \xrightarrow{\rho} & M \otimes C \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle 1\otimes\Delta} \\ M \otimes C & \xrightarrow{\rho\otimes 1} & M \otimes C \otimes C \end{array}$$

and

$$\begin{array}{ccc} M & & \\ {\scriptstyle\rho}\downarrow & \searrow{\scriptstyle\cong} & \\ M \otimes C & \xrightarrow{1\otimes\epsilon} & M \otimes \mathbb{K} \end{array}$$

Let (M, ρ_M) and (N, ρ_N) be two comodules. A $\mathbb{K}$-linear map $f : M \to N$ is called a morphism of comodules if the following diagram is commutative:

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ {\scriptstyle \rho_M}\downarrow & & \downarrow{\scriptstyle \rho_N} \\ M \otimes C & \xrightarrow{\; f \otimes 1 \;} & N \otimes C \end{array}$$

Notation: $\mathbf{M}^C$

The cofree C-comodule:

If (C, Δ, ϵ) is a coalgebra and V a $\mathbb{K}-$vector space then $V \otimes C$ becomes a right C-comodule with

$$\rho = 1 \otimes \Delta : V \otimes C \to V \otimes C \otimes C$$

Cosemisimple coalgebras, completely reducible comodules

Definition 13. A coalgebra C is called *simple* if $C \neq 0$ and it has no proper subcoalgebras. A coalgebra C is called *cosemisimple* if it is a direct sum of simple subcoalgebras.

A comodule C is said to be *irreducible* if $V \neq 0$ and it has no proper subcomodules. A comodule is called *completely reducible* if $V = 0$ or V is a direct sum of irreducible subcomodules.

Proposition 3. • Every simple coalgebra is finite dimensional.

• Every coalgebra is sum of finite dimensional subcoalgebras.

Proposition 4. For a given coalgebra C the following assertions are equivalent:

a) C is cosemisimple

b) C is sum of simple subcoalgebras

c) If D is any subcoalgebra of C then there exists a subcoalgebra E of C such that $C = D \oplus E$

d) Every subcoalgebra of C is cosemisimple

e) Every finite dimensional subcoalgebra of C is cosemisimple

Proposition 5.

- Every irreducible comodule is finite dimensional.

- Every comodule is sum of finite dimensional subcomodules.

Proposition 6. For a given comodule V the following assertions are equivalent:

a) V is completely reducible

b) V is sum of irreducible subcomodules

c) If W is any subcomodule of V then there exists a subcomodule Z of C such that $V = W \oplus Z$

d) Every subcomodule of V is completely reducible

e) Every finite dimensional subcomodule of V is completely reducible

Theorem 1. *Given a coalgebra C the following are equivalent:*

- *C is cosemisimple*
- *every C comodule is completly reducible*

Indexed categories by coalgebras

We consider an C-*indexed category* of comodules

$$\Phi : \mathbf{Coalg}^{op} \to \mathbf{Cat}$$

given by $\Phi(C) =^C \mathbf{M}$.

Notation: ${}^C\mathbf{M} = Vect^C$ the category of left C-comodules indexed by the coalgebra C.

Finite products and equalizers exist in $Vect^C$ and are those of vector spaces.

Let $\phi : D \to C$ be a morphism of coalgebras, we consider the functor $\phi^* : Vect^C \to Vect^D$ determined by the following equalizer:

$$\begin{array}{ccccc} E & \xrightarrow{e} & D \otimes M & \xrightarrow{\Delta \otimes 1_M} & D \otimes D \otimes M \\ & & & {\scriptstyle 1_D \otimes \rho} \searrow & \downarrow {\scriptstyle 1_D \otimes \phi \otimes 1_M} \\ & & & & D \otimes C \otimes M \end{array}$$

i.e., $\phi^*(M, \rho) = E$ on object and

by the universal property of equalizer on arrows, in which all the coactions considered above come from the cofree comodule structure except for E which has the restriction of the cofree coaction of $D \otimes M$.

So we define a pseudofunctor: $\Phi : \mathbf{Coalg}^{op} \to \mathbf{Cat}$ given by $\Phi(C) = Vect^C$, $\Phi(\phi) = \phi^*$ i.e., $(\phi\psi)^* \cong \psi^*\phi^*$, $1_C^* \cong 1_{Vect^C}$.

For each $\phi : C \to D$, $\phi^* : Vect^D \to Vect^C$ has a left adjoint $\sum_\phi \vdash \phi^*$; $\sum_\phi : Vect^C \to Vect^D$ given by $\sum_\phi(V, v) = (V, (\phi \otimes id_V)v)$.

$$V \xrightarrow{v} C \otimes V \xrightarrow{\phi \otimes id_V} D \otimes V.$$

Proposition 7. Let $\pi_C : D\otimes C \to C$, $\pi_D : D\otimes C \to D$ be projection maps in the category $\mathbf{Coalg}$. Then $\pi_C^* : Vect^C \to Vect^{D\otimes C}$ and $\pi_D^* : Vect^D \to Vect^{D\otimes C}$ preserves coequalizers.

Also we have explicit formulas:

$$\pi_C^*(M,\rho) = (D \otimes M, \rho')$$

where ρ' is

$$D \otimes M \overset{\Delta\otimes\rho}{\to} D \otimes D \otimes C \otimes M \overset{\sigma}{\to} D \otimes C \otimes D \otimes M$$

and analogously π_C^*.

For every $\phi : C \to D$ the functor $\phi^* : Vect^D \to Vect^C$ preserves coproducts, i.e.,

$$\phi^*(\oplus_{i\in I}(C_i, \rho_i)) = \oplus_{i\in I}\phi^*(C_i, \rho_i)$$

for arbitrary I but in general do not preserve coequalizers.

The last proposition implies that

$$\pi_C^* \quad \pi_D^*$$

preserves colimits and by special adjoint functor theorem has a right adjoint.

$Vect^C$ **symmetric monoidal closed category**

Lemma 1. *If C is a cocommutative coalgebra, the category $Vect^C$ is a symmetric monoidal category.*

The tensor in $Vect^C$ is defined as follows:

take C-comodules $(V, v), (W, w)$ and consider the following equalizer:

$$E \xrightarrow{\quad e \quad} V \otimes W \underset{\tau v \otimes id_W}{\overset{id_V \otimes w}{\rightrightarrows}} V \otimes C \otimes W \tag{1}$$

i.e., $E = (V, v) \otimes^C (W, w)$ and the coaction is given by the universal property where

$(V \otimes W, v \otimes id_W)$ and $(V \otimes C \otimes W, v \otimes id_C \otimes id_W)$.

$$\begin{array}{ccccc}
E & \xrightarrow{e} & V \otimes W & \underset{\tau v \otimes id_W}{\overset{id_V \otimes w}{\rightrightarrows}} & V \otimes C \otimes W \\
\downarrow{\scriptstyle \rho_{V \otimes W}} & & \downarrow{\scriptstyle v \otimes 1} & & \downarrow{\scriptstyle v \otimes 1} \\
C \otimes E & \xrightarrow{id_C \otimes e} & C \otimes V \otimes W & \underset{id_C \otimes \tau v \otimes id_W}{\overset{id_C \otimes id_V \otimes w}{\rightrightarrows}} & C \otimes V \otimes C \otimes W
\end{array} \qquad (2)$$

since $C \otimes -$ preserves equalizers and the unit is given by

$I = (C, \Delta_C)$.

Lemma 2. *If C is a cocommutative coalgebra, the monoidal category $(Vect^C, \otimes^C, C)$ is closed if and only if C is cosemisimple.*

$Coalg^C$ **cartesian category**

The category $Coalg^C = Coalg/C$ (slice category) defined as follows:

- objects are morphisms of coalgebras with cocommutative codomain in C; we denote by (ϕ) the morphism of coalgebras $\phi : D \to C$ when it is thought as an object in $Coalg^C$,

- if $\phi : D \to C$ and $\psi : E \to C$ are morphisms of coalgebras, morphisms $f : (\phi) \to (\psi)$ correspond to coalgebra morphisms $f : D \to E$ such that $\psi \circ f = \phi$;

Lemma 3. *If C is a cocommutative coalgebra, the category $Coalg^C$ is a cartesian category.*

Proof. The existence of finite products and equalizers in $Coalg$ guarantees the existence of pullbacks in this category, that induce a cartesian structure on $Coalg^C$.
We have that $(\phi_1) \times (\phi_2) = (\phi)$, where ϕ is defined by the following pullback in $Coalg$:

$$\begin{array}{ccc} D & \xrightarrow{u} & D_1 \\ {\scriptstyle v}\downarrow & \searrow^{\phi} & \downarrow{\scriptstyle \phi_1} \\ D_2 & \xrightarrow[\phi_2]{} & C. \end{array}$$

Moreover, the unit object is (id_C).

□

Monoidal adjunction: $(U^C, m) \dashv (R^C, n)$

The functor $U^C : Coalg^C \to Vect^C$ takes the object (ϕ), i.e., $\phi : D \to C$ to the comodule (D, d) where $d : D \to D \otimes C$ is the coaction defined by $d = (\phi \otimes id_D) \circ \Delta_D$ admits a right adjoint: $U^C \dashv R^C$.

Lemma 4. *The functor* $U^C : Coalg^C \to Vect^C$ *is strong monoidal.*

Proof. It is clear that $U^C((id_C)) = (C, \Delta)$, so U^C preserves the units.

We will prove now that

$$U^C((\phi) \times (\psi)) = U^C(\phi) \otimes^C U^C(\psi)$$

.

Take $\phi_1 : (D_1, \Delta_1, \varepsilon_1) \to (C, \Delta, \varepsilon)$, and

$\phi_2 : (D_2, \Delta_2, \varepsilon_2) \to (C, \Delta_C, \varepsilon_C)$ two morphisms of coalgebras.

We recall the diagram defining the product $(\phi) = (\phi_1) \times (\phi_2)$:

$$\begin{array}{ccc} D & \xrightarrow{u} & D_1 \\ {\scriptstyle v}\downarrow & \searrow^{\phi} & \downarrow{\scriptstyle \phi_1} \\ D_2 & \xrightarrow[\phi_2]{} & C. \end{array}$$

Note that $U^C(D_i) = (D_i, d_i)$ for $i = 1, 2$ and $U^C(D) = (D, d)$ where $d = (\phi \otimes id_D) \circ \Delta$, $d_1 = (\phi_1 \otimes id_{D_1}) \circ \Delta_1$, $d_2 = (\phi_2 \otimes id_{D_2}) \circ \Delta_2$.

We will prove that $(D, d) = (D_1, d_1) \otimes^C (D_2, d_2)$, in other words that D-with a suitable morphism d- is the equalizer in $Vect$ of the following parallel pair and that d is effectively $\rho_{D_1 \otimes^C D_2}$ (with the notation of Lemma 2), i.e.,

$$D \xrightarrow{e} D_1 \otimes D_2 \underset{\tau d_2 \otimes id_{D_2}}{\overset{id_{D_1} \otimes d_1}{\rightrightarrows}} D_1 \otimes C \otimes D_2 \qquad (3)$$

Idea of the proof:

1-First observe that the parallel pair above can be though in $Coalg$. We prove first that the coalgebra D-with the morphism of coalgebras $(u \otimes v) \circ \Delta : D \to D_1 \otimes D_2$ is the equalizer in $Coalg$.

2-Now, as U preserves equalizers of the coreflexive pairs, we have that $\{D, (u \otimes v)\Delta\}$ is the equalizer in $Vect$ of the parallel pair above. (Note that the pair is coreflexive for $id_{D_1} \otimes \varepsilon_C \otimes id_{D_2}$ is a common retraction in $Coalg$.)

3-It is easy to prove that d is the desired coaction, i.e. that the following diagram commutes:

$$\begin{array}{ccc}
D & \xrightarrow{(u\otimes v)\circ\Delta} & D_1 \otimes D_2 \\
\downarrow d & & \downarrow d_1\otimes id_{D_2} \\
C \otimes D & \xrightarrow{id_C\otimes((u\otimes v)\circ\Delta)} & C \otimes D_1 \otimes D_2
\end{array}$$

$\square$

ϕ^* has left adjoint $\sum_\phi$. But in general is not the case that ϕ^* and $- \otimes^C A$ preserve coequalizers.

We want to study conditions to obtain right adjoints:

$\phi^* \dashv \Pi_\phi$ and $- \otimes^C A \dashv hom^C(A, -)$

Definition 14. we said that a C-comodule (V, ρ) is *coflat* when the functor

$$- \otimes^C V : Vect^C \to Vect^C$$

preserves epis.

Proposition 8. Let (V, ρ) be a C-comodule. The following propositions are equivalent:

- (V, ρ) is coflat.

- $V \otimes^C - : Vect^C \to Vect^C$ has a right adjoint $hom^C(V, -) : Vect^C \to Vect^C$.

Proposition 9. Let $\phi : V \to W$ be a coalgebra map. The following propositions are equivalent:

- $(V, (id \otimes \phi)\Delta)$ C-comodule is coflat.
- $\phi^* : Vect^W \to Vect^V$ has a right adjoint $\Pi_\phi : Vect^V \to Vect^W$.

Beck condition. It turns out that since we have $\sum_\phi \dashv \phi^* \dashv \Pi_\phi$ and $\sum_\phi$ satisfies that condition then Π_ϕ also satisfies Beck condition whenever it exists:

$$\begin{array}{ccc} A & \xrightarrow{\vartheta} & B \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\psi} \\ C & \xrightarrow[\eta]{} & D \end{array}$$

is a pullback then

$$\begin{array}{ccc} Vect^B & \xrightarrow{\vartheta^*} & Vect^A \\ \downarrow \Pi_\psi & & \downarrow \Pi_\phi \\ Vect^D & \xrightarrow[\eta^*]{} & Vect^C \end{array}$$

commutes.