Changing the structure in implicative algebras

Realizabilidad en Uruguay
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Abstract

We explain Streicher’s construction of categorical models of classical realizability in terms of a change of the structure in an implicative algebra with a closure operator. We show how to perform a similar construction using another closure operator that produces a different categorical model that has the advantage of being –at a difference with Streicher’s constructio– an implicative algebra. Some of the results I will present appeared in the ArXiv and others are being currently developped.

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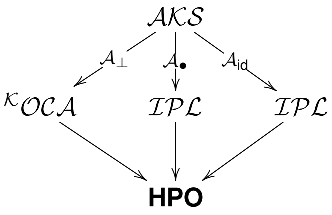
We explain Streicher’s construction of categorical models of classical realizability in terms of a change of the structure in an implicative algebra with a closure operator. We show how to perform a similar construction using another closure operator that produces a different categorical model that has the advantage of being –at a difference with Streicher’s constructio– an implicative algebra. Some of the results I will present appeared in the ArXiv and others are being currently developped.

Main diagram and nomenclature

Main diagram



Nomenclature

- $\mathcal{AKS}$: $\longleftarrow$ Abstract Krivine Structure,
- $^{\mathcal{K}}\mathcal{OCA}$: $\longleftarrow$ K, ordered combinatory algebra,
- $\mathcal{IPL}$: $\longleftarrow$ Implicative algebra,
- **HPO**: $\longleftarrow$ Heyting preorder.

Introductory words

- Until 2013 –with the work of Streicher– it was not easy to see how Krivine’s work on classical realizability, could fit into the structured categorical approach initiated by Hyland in 1982.
- Streicher’s proposal to fill the gap followed the standard method consisting in the construction of a realizability tripos followed with the tripos–to–topos construction.
- This construction –as shown by Mauricio– consists in the composition of the two arrows on the left (he did not construct the factors but the composition), and he produced from an abstract Krivine structure a Heyting preorder (**HPO**).
- We will consider the pros and cons of this construction and we will compare it with the one in the center of the diagram –the one based upon $\mathcal{A}_\bullet$ that was developed recently from work within the group (M. Guillermo, O. Malherbe,WF, of course with the help of Alexandre).
- I will present the constructions of this diagram as a process of *change of implication*, applying to the rightmost diagram two different closure operators to produce the change.

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1 Introduction

2 Implicative algebras, changing the implication
- Interior and closure operators
- Use of the interior operator to change the structure

3 From abstract Krivine structures to structures of “implicative nature”
- Krivine’s construction; Streicher’s construction
- Dealing with the lack of a full adjunction

4 $\mathcal{OCA}$s and triposes

Interior operators

Basic definitions

Let $\mathcal{A} = (A, \leq, \to)$ be an implicative structure; an interior operator is a map $\iota : A \to A$ such that:

- ι is monotonic.
- If $a \in A$, $\iota(a) \leq a$;
- $\iota^2 = \iota$.

Call $A_\iota = \{a \in A : \iota(a) = a\} = \iota(A)$ the ι–open elements of A.

- If the map satisfies $\iota(\curlywedge_j a_j) = \curlywedge_j \iota(a_j)$ for all $\{a_j : j \in I\} \subseteq A$, it is said to be an Alexandroff interior operator or an A–interior operator.

Associated closure

- Assume that $\iota : A \to A$ is an A–interior operator, define $c_\iota : A \to A$ as: $c_\iota(a) = \curlywedge\{b \in A_\iota : a \leq b\}$.
- c_ι is a closure operator –i.e. an interior operator for the opposite order $\geq$.
- The set of closed elements for c_ι coincides with A_ι.

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Use interior operators to change implication

Basic properties of A–operators

- $\{a_j : j \in I\} \subseteq A_\iota$ then: $\curlywedge_j a_j \in A_\iota$; so that A_ι es inf complete.
- $(A_\iota, \subseteq, \curlywedge)$ is a complete meet semilattice.
- If $a, b \in A_\iota$, then $a \to_\iota b = \iota(a \to b)$ is an implicative structure in $(A_\iota, \subseteq, \curlywedge)$ equipped with the order of A restricted.

Proof.

Assume that $a \in A_\iota$, $B \subseteq A_\iota$, then $a \to_\iota \curlywedge B = \iota(a \to \curlywedge B) = \iota\left(\curlywedge_{b \in B}(a \to b)\right) = \curlywedge_{b \in B} \iota(a \to b) = \curlywedge_{b \in B}(a \to_\iota b)$. □

The above is not true for a general interior operator (i.e. a *not Alexandroff* closure operator).

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The application and the adjunction

The *application* associated to the implication

If $(A, \leq, \to)$ is an implicative algebra, the associated *application* is $\circ_\to : A \times A \to A$ defined as:

$$a \ \circ_\to \ b = \bigwedge \{c : a \leq b \to c\},$$

and this implies (in fact it is equivalent to the fact) that $\circ_\to$ and $\to$ are adjoints, i.e.

$$a \ \circ_\to \ b \leq c \text{ if and only if } a \leq b \to c,$$

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Main property

Theorem

Let $\mathcal{A} = (A, \leq, \rightarrow)$, be an implicative algebra and $\circ_\rightarrow$, ι and c_ι as above. If we change the implication (i.e. consider the implicative algebra $(A_\iota, \leq, \curlywedge, \rightarrow_\iota)$) where $\rightarrow_\iota := \iota \rightarrow$, then the corresponding application is:

$$A_\iota \times A_\iota \xrightarrow{\circ_\rightarrow} A_\iota \xrightarrow{c_\iota} A_\iota,$$

i.e. $\forall a, b \in A_\iota$: $\curlywedge\{d \in A_\iota : a \leq b \rightarrow_\iota d\} = c_\iota\left(\curlywedge\{d \in A : a \leq b \rightarrow d\}\right)$.

Proof.

We have that:

- $\curlywedge\{d \in A_\iota : a \leq b \rightarrow_\iota d\} = \curlywedge\{d \in A_\iota : a \leq \iota(b \rightarrow d)\} = \curlywedge\{d \in A_\iota : a \leq b \rightarrow d\} = \curlywedge\{d \in A_\iota : a \circ_\rightarrow b \leq d\} = c_\iota(a \circ_\rightarrow b)$
- For the second equality use that $a \in A_\iota, e \in A$, $a \leq \iota(e) \Leftrightarrow a \leq e$, and for the fifth, the definition of c_ι.

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- For the second equality use that $a \in A_\iota, e \in A$, $a \leq \iota(e) \Leftrightarrow a \leq e$, and for the fifth, the definition of c_ι.

Main property

Theorem

Let $\mathcal{A} = (A, \leq, \to)$, be an implicative algebra and $\circ_\to$, ι and c_ι as above. If we change the implication (i.e. consider the implicative algebra $(A_\iota, \leq, \curlywedge, \to_\iota)$) where $\to_\iota := \iota \to$, then the corresponding application is:

$$A_\iota \times A_\iota \xrightarrow{\circ_\to} A_\iota \xrightarrow{c_\iota} A_\iota,$$

i.e. $\forall a, b \in A_\iota$: $\curlywedge\{d \in A_\iota : a \leq b \to_\iota d\} = c_\iota(\curlywedge\{d \in A : a \leq b \to d\})$.

Proof.

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Summary

Changing the structure with ι

Original structure	*New structure*
$a, b \in A\,,\ \leq\,,\ \inf = \curlywedge$	$a, b \in A_\iota\,,\ \leq\,,\ \inf = \curlywedge$
$a \to b$	$a \to_\iota b = \iota(a \to b)$
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Abstract Krivine structures

$\mathcal{AKS}$

$\mathcal{K} = (\Lambda, \Pi, \perp\!\!\!\perp, \text{push}, \text{app}, \text{store}, \text{QP}, \kappa, \text{s}, \text{cc}) \in \mathcal{AKS}$.

$\perp\!\!\!\perp \subseteq \Lambda \times \Pi$

$t \perp \pi$ means $(t, \pi) \in \perp\!\!\!\perp$

push : $\Lambda \times \Pi \to \Pi$

app : $\Lambda \times \Lambda \to \Lambda$

store : $\Pi \to \Lambda$

$\text{QP} \subseteq \Lambda$

$\kappa, \text{s}, \text{cc} \in \text{QP}$

push$(t, \pi) := t \cdot \pi$; app$(t, \ell) := t\ell$

store$(\pi) := k_\pi$

QP is closed under app

If $t \perp \ell \cdot \pi$ then $t\ell \perp \pi$

If $t \perp \pi$ then $\kappa \perp t \cdot \ell \cdot \pi$

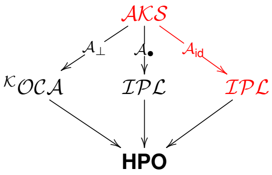
If $(tu)\ell u \perp \pi$ then $\text{s} \perp t \cdot \ell \cdot u \cdot \pi$

If $t \perp k_\pi \cdot \pi$ then $\text{cc} \perp t \cdot \pi$

If $t \perp \pi$ then $k_\pi \perp t \cdot \rho$

1. Introduction
2. Implicative algebras, changing the implication
   - Interior and closure operators
   - Use of the interior operator to change the structure
3. From abstract Krivine structures to structures of “implicative nature”
   - Krivine’s construction; Streicher’s construction
   - Dealing with the lack of a full adjunction
4. $\mathcal{OCA}$s and triposes

Towards *implicative* structures I



The construction $\mathcal{A}_{\text{id}}$

$\mathcal{A}_{\text{id}}(\mathcal{K}) = (\mathcal{P}(\Pi), \supseteq, \curlywedge, \to, \Phi)$ is an *implicative algebra*.

- $P \subseteq \Pi$; ${}^\perp P := \{t \in \Lambda : (t, P) \subseteq \perp\!\!\!\perp\}$ $\longleftarrow$ the pole $\perp\!\!\!\perp$
- $L \subseteq \Lambda$; $L^\perp := \{\pi \in \Pi : (L, \pi) \subseteq \perp\!\!\!\perp\}$ $\longleftarrow$ the pole $\perp\!\!\!\perp$
- Given $P, Q \in \mathcal{P}(\Pi)$ define: $P \curlywedge Q := P \cup Q$
- Given $\chi \subseteq \mathcal{P}(\Pi)$ define $\bigcurlywedge \chi := \bigcup \chi$.
- Given $P, Q \in \mathcal{P}(\Pi)$ define
  $P \to Q := \text{push}({}^\perp P, Q) = \{t \cdot \pi : t \in {}^\perp P, \pi \in Q\} \subseteq \Pi$ $\leftarrow$ the push : $\Lambda \times \Pi \to \Pi$
- The *filter* or separator $\Phi = \{P \subseteq \Pi : \exists t \in \text{QP}, t \perp P\}$.

Remark

Let us compute the application map associated to the implication

$$P \circ_{\rightarrow} Q := \bigcup \{R : P \supseteq Q^{\perp} \cdot R\},$$

(c.f. previous section and recall that $Q \rightarrow R = Q^{\perp} \cdot R$). It is clear that this coincides with Streicher's:

$$P \circ Q := \{\pi \in \Pi : P \supseteq {}^{\perp}Q \cdot \pi\}.$$

Full adjunction

Being an implicative algebra and as $\circ = \circ_{\rightarrow}$ the following full adjunction holds:

$$P \leq Q \rightarrow R \quad \text{if and only if} \quad P \circ Q \leq R.$$

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Towards *implicative* structures II

The construction $\mathcal{A}_\perp$ (T. Streicher–2013)

$\mathcal{A}_\perp(\mathcal{K}) = (\mathcal{P}_\perp(\Pi), \supseteq, \wedge_\perp, \to_\perp, \circ_\perp)$ is a ${}^{\mathcal{K}}\mathcal{OCA}$ –not implicative.

- $\iota(P) = \overline{P} := ({}^\perp P)^\perp$.
- $\mathcal{P}(\Pi)_\iota = \mathcal{P}_\perp(\Pi) := \{P \subseteq \Pi \,|\overline{P} = P\}$.
- Given $P, Q \in \mathcal{P}_\perp(\Pi)$ define $P \wedge_\perp Q := (P \cup Q)^-$.
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- Hence $(\mathcal{P}_\perp(\Pi), \supseteq, \curlywedge_\perp)$ is an inf complete semilattice.
- Given $P, Q \in \mathcal{P}(\Pi)$ define

$$P \to_\perp Q := (P \to Q)^- \qquad P \circ_\perp Q := (P \circ Q)^-$$

- We take as separator (called filter in this context) the intersection $\Phi_\perp = \Phi \cap \mathcal{P}_\perp(\Pi)$.

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Towards implicative structures III

Summary, $\mathcal{K} \in \mathcal{AKS}$

$\mathcal{A}_{\mathrm{id}}$ (Krivine)	$\mathcal{A}_{\perp}$ (Streicher)
$P, Q \in \mathcal{P}(\Pi)$	$P, Q \in \mathcal{P}_{\perp}(\Pi)$
$P \to Q$	$P \to_{\perp} Q = (P \to Q)^{-}$
$P \circ Q$	$P \circ_{\perp} Q = (P \circ Q)^{-}$
$P \supseteq Q \to R$ iff $P \circ Q \supseteq R$	if $P \supseteq Q \to_{\perp} R$ then $P \circ_{\perp} Q \supseteq R$

The operations given by Streicher do not have behave well with respect to the adjunction relation because the closure operator is not Alexandroff.

Towards implicative structures IV

Proving the half adjunction

$P, Q \in \mathcal{P}_\perp(\Pi)$.

- $P \leq Q \to_\perp R = \iota(Q \to R)$ if and only if $P \leq Q \to R$ (basic property of the interior operator).
- $P \leq Q \to R$ if and only if $P \circ Q \leq R$ (basic adjunction property for $\mathcal{P}(\Pi)$).
- If $P \circ Q \leq R$ then $P \circ_\iota Q = \iota(P \circ Q) \leq \iota(R) = R$ (using the monotony of the interior operator).

The last part of the argument cannot be reversed!!

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Streicher’s solution: the adjunctor

The adjunctor for $\to_\perp$, $\circ_\perp$

- If $P \supseteq Q \to_\perp R$ then $P \circ_\perp Q \supseteq R$. ✓
- From the basic elements K and S we build an element $\mathrm{E} \in \Lambda$ with the property: $tu \perp \pi$ implies that $\mathrm{E} \perp t \cdot u \cdot \pi$. Define $E := \{\mathrm{E}\}^\perp \in \mathcal{P}_\perp(\Pi)$.
- If $P \circ_\perp Q \supseteq R$ then $E \circ_\perp P \supseteq Q \to_\perp R$. ✓

Adjunctor in one line

$$(P \supseteq Q \to_\perp R) \Rightarrow (P \circ_\perp Q \supseteq R) \Rightarrow (E \circ_\perp P \supseteq Q \to_\perp R)$$

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${}^{\mathcal{K}}\mathcal{OCA}$s: Why do we need them?

Motivation

The construction $\mathcal{A}_{\text{id}}$ needs only one operator –as it is implicative– Streicher's needs two –as it is not–, that is the motivation (post factum) we had to define ${}^{\mathcal{K}}\mathcal{OCA}$s.

The definition of ${}^{\mathcal{K}}\mathcal{OCA}$

A ${}^{\mathcal{K}}\mathcal{OCA}$ –a oca with adjunctor– has the following ingredientes

- $(A, \leq, \inf)$ an inf complete partially ordered set.
- $\rightarrow, \circ : A^2 \rightarrow A$ two maps with the same monotony conditions considered before.
- $\Phi \subseteq A$ a *filter* that is closed by application and upwards closed w.r.t. the order.
- Three elements $\mathrm{K}, \mathrm{S}, \mathrm{E} \in \Phi$ with the same properties than the ones considered before and one more: $\forall a, b, c \in A$:

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- Three elements $\text{K}, \text{S}, \text{E} \in \Phi$ with the same properties than the ones considered before and one more: $\forall a, b, c \in A$:

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In the case that the adjunctor does not appear i.e. if $a \circ b \leq c \Rightarrow a \leq b \to c$, we have an *implicative algebra*.

${}^{\mathcal{K}}\mathcal{OCA}$s: Why do we need them?

Motivation

The construction $\mathcal{A}_{\text{id}}$ needs only one operator –as it is implicative– Streicher's needs two –as it is not–, that is the motivation (post factum) we had to define ${}^{\mathcal{K}}\mathcal{OCA}$s.

The definition of ${}^{\mathcal{K}}\mathcal{OCA}$

A ${}^{\mathcal{K}}\mathcal{OCA}$ –a oca with adjunctor– has the following ingredientes

- $(A, \leq, \inf)$ an inf complete partially ordered set.
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Another solution: the Alexandroff approximation

The concept of A–approximation

- $(A, \leq)$ a meet complete semilattice, and $\mathcal{I}(A)(\mathcal{I}_\infty(A))$ the set of its interior operators (A–interior operators), for $\iota, \kappa \in \mathcal{I}(A)$ we say that $\iota \leq \kappa$ if for all $a \in A$, $\iota(a) \leq \kappa(a)$.
- An operator ι is A–approximable if the non empty set $\{\kappa \in \mathcal{I}_\infty(A) : \iota \leq \kappa\}$ has a minimal element: ι_∞.
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- $P \mapsto ({}^{\perp}P)^{\perp} = \overline{P} = \iota(P)$ is an interior operator (not Alexandroff). Change $\to$ to $\to_{\iota} = \iota \to$, and $\circ$ to $\circ_{\iota} = \iota\circ$. The adjunction property fails because we change the operations both with the interior operator (double perpendicularity).
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Summary

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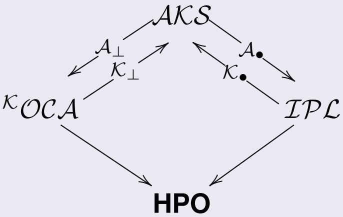
Back to the main diagram

Going back in the constructions



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With the purpose to further the algebraization program and take $\mathcal{OCA}$s or more specifically implicative algebras as a foundational basis for classical realizability and make sure that we do not loose information, we construct maps going back in the diagram.

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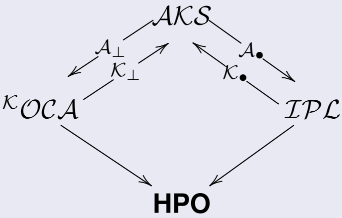

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We can forget about the $\mathcal{AKS}$

From $\mathcal{OCA}$s to $\mathcal{AKS}$

We describe the construction $\mathcal{K}_\bullet : \mathcal{IPL} \to \mathcal{AKS}$.

$$\mathcal{A} = (A, \leq, \mathrm{app}, \mathrm{imp}, \mathsf{k}, \mathsf{s}, \Phi) \mapsto \mathcal{K}_\bullet(\mathcal{A}) = (\Lambda, \Pi, \perp\!\!\!\perp, \mathrm{app}, \mathrm{push}, \mathsf{K}, \mathsf{S}, \mathrm{QP})$$

as follows.

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3. $\mathrm{app}(s,t) := st \quad , \quad \mathrm{push}(s,\pi) := \mathrm{imp}(s,\pi) = s \to \pi$;
4. $\mathsf{K} := \mathsf{k} \quad , \quad \mathsf{S} := \mathsf{s}$;
5. $\mathrm{QP} := \Phi$.

We can forget about the $\mathcal{AKS}$

From $\mathcal{OCA}$s to $\mathcal{AKS}$

We describe the construction $\mathcal{K}_\bullet : \mathcal{IPL} \to \mathcal{AKS}$.

$$\mathcal{A} = (A, \leq, \mathrm{app}, \mathrm{imp}, \mathsf{k}, \mathsf{s}, \Phi) \mapsto \mathcal{K}_\bullet(\mathcal{A}) = (\Lambda, \Pi, \perp\!\!\!\perp, \mathrm{app}, \mathrm{push}, \mathsf{K}, \mathsf{S}, \mathrm{QP})$$

as follows.

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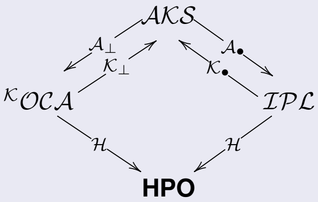
Back to the main diagram

Forgetting the $\mathcal{AKS}$



There is no need for the $\mathcal{AKS}$ when we apply $\mathcal{H}$.

Assume that $\mathcal{A}$ is a $^{\mathcal{K}}\mathcal{OCA}$ or an implicative algebra.

1. If $\mathcal{A}$ is a $^{\mathcal{K}}\mathcal{OCA}$, then $\mathcal{A}$ and $\mathcal{A}_\perp(\mathcal{K}_\perp(\mathcal{A}))$ are isomorphic. Hence, they produce isomorphic **HPO**s and triposes ... and topoi.
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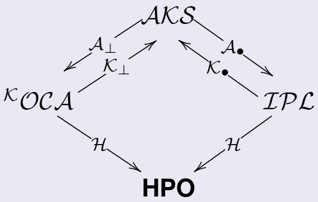
Back to the main diagram

Forgetting the $\mathcal{AKS}$



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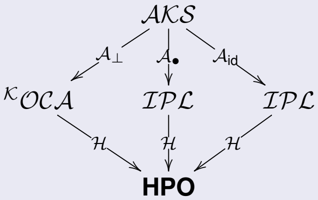
The last piece of the construction

We want to show that we do not loose any information by changing the implication as we have been doing.

The final comparison



- Assume that $\mathcal{K}$ is an abstract Krivine structure: then the inclusions

$$\mathcal{H}(\mathcal{A}_\perp(\mathcal{K})) \subseteq \mathcal{H}(\mathcal{A}_\bullet(\mathcal{K})) \subseteq \mathcal{H}(\mathcal{A}(\mathcal{K})),$$

  are equivalences of preorders.

Thank you for your attention