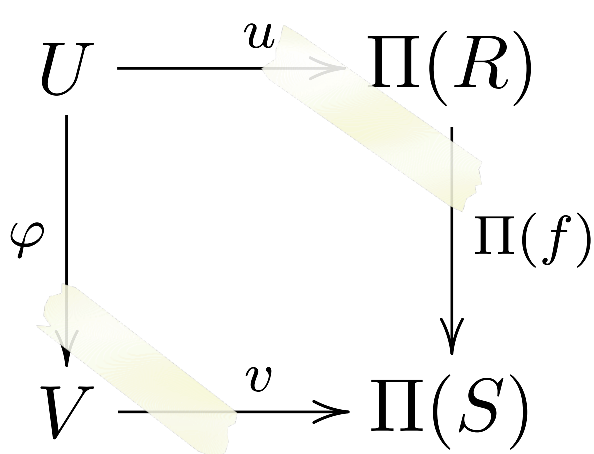
Double-glueing and Linear Logic

beyond LL: enriched models, polarization and games

Pierre-Marie Pédrot

MPRI

M2 Research Internship: March–August 2011



Plume, LIP

Supervisor: Olivier Laurent (Plume)

La logique mène à tout, à condition d'en sortir.

ALPHONSE ALLAIS,
à propos de la correspondance de Curry-Howard.

Acknowledgements

In the first place, I would like to warmly thank Olivier Laurent, for having provided me with an interesting internship subject, as well as for his helpful pieces of advice.

I would also like to thank my fellow co-office colleagues, for having borne with me and my frequent psychotic crises about linear logic. This also includes Alexandre Miquel, who saved the day (and my thesis).

I would like more generally to thank everyone from the lab, from the playful Daniel Hirschkoff, to the vindictive Barbara Petit (whose *pot de thèse* was peculiarly remarkable), as well as many others.

Finally, I would like to thank all those who entertained me during my internship: the vast range of mailing-lists, Branquignoles, Baaah, Coqloc and others, which flooded my mailbox on a daily basis; the Erasmus students who tore my liver down, gatecrashed my seminar and tricked me into going to a Third World country; the few efficient RF who filled the fridges of the *foyer* up; and the allmighty and blood-thirsty god IMPEGA who benevolently watched over my miserable and filthy being during the whole internship.

Introduction

Linear logic, which was designed by Girard [7] at the end of the 80's as a refinement of both intuitionistic and classical logic, is a logic that deals with *resources*. Indeed, in the general case, a formula can be neither tossed away nor duplicated, contrary to more usual logical systems. Resources must be used *linearly*, whence its name. Historically, linear logic was born upon the fertile soil of model analysis: it was conceived from the famous decomposition $A \Rightarrow B := !A \multimap B$ in what was about to become coherent spaces.

Even though models of the perfect fragment of linear logic (another name for the MALL fragment) where formalized through category theory in no time, exponentials revealed to be quite cumbersome. Actually, the first attempts even turned out to be unsound. There is still today different paradigms which are not strictly equivalent [16].

An important property of linear logic, which also appears through intuitionistic logic in a fossil state, comes from its polarization. This can be summarized from a proof-search point of view by stating that some connectives are *invertible*. Polarization led to important semantical discoveries.

In 2003, Hyland and Schalk [9] described a categorical construction inspired from realizability and games, hereafter called double-glueing[1]. Double-glueing takes a model of (a fragment of) LL and returns a more fine-grained one. The idea underlying double-glueing consists in refining types from a given model with respect to well-behavedness towards a dual set of opponent strategies. This notion of dynamic types is at work in ludics [8]. Dynamic types are not to be understood as in dynamic typing[2]: it rather means that types are defined *a posteriori* through a dynamic process of *orthogonality*. Similarly to ludics, the worse the base model, the better the result will be.

Interestingly enough, it turned out that double-glueing could formalize historical models of LL and factorize them in a nifty way. On the other hand, this construction can be thought as a model-factory, able to design models well-fitted for a peculiar task. This is akin to the choice of a particular pole in reducibility candidate proofs.

We will define in this report a generalization of double-glueing to a more general class of models, so-called enriched models. We shall also underline the fact that double-glueing *creates* polarization out of an unpolarized model, which is something quite surprising. We also give a whole range of results concerning particularities of double-glueing, from focussed orthogonalities to non-uniform exponentials.

[1] There also exists a simple-glueing construction, which is similar to intuitionistic realizability.

[2] As in the Python language, for example.

1 Double-glueing: generalities

We will not be that precise in this section, which is to be taken as a gentle introduction to the general ideas lying beneath double-glueing.

Double-glueing is deeply rooted into realizability, though it is also a purely categorical construct. It also shares many common points with famous models of linear logic, as a lot of them are instances of double-glueing. As such, ludics is not alien to it, even though the latter is a bit too untyped.

1.1 Generic process

Let $\mathcal{M}$ be a categorical model of whatever the reader can imagine. Double-glueing goes as follows:

$$\mathcal{M} \longrightarrow \mathbb{G}(\mathcal{M}) \begin{array}{l} \dashrightarrow \mathbb{S}(\mathcal{M}) \\ \dashrightarrow \mathbb{T}(\mathcal{M}) \end{array}$$

The model $\mathcal{M}$ is almost irrelevant to the first step, which consists in making the objects of $\mathcal{M}$ explode.

Definition 1.1. We say that $\mathcal{M}$ is equipped with an abstract notion of (typed) proofs and counter-proofs[3] whenever for each object R there exist a set of R-proofs and a set of R-counter-proofs such that:

- for any $f : \mathcal{M}(R, S)$ and any proof $\pi \Vdash^{p} R$, there exists a composite proof $\pi \triangleright f \Vdash^{p} S$
- for any $f : \mathcal{M}(R, S)$ and any counter-proof $\xi \Vdash^{o} S$, there exists a composite counter-proof $f \triangleleft \xi \Vdash^{o} R$

which are both subject to compatibility with categorical composition, that is:

- For any $f : \mathcal{M}(R, S)$, $g : \mathcal{M}(S, T)$ and $\pi \Vdash^{p} R$, we have $\pi \triangleright (f; g) = (\pi \triangleright f) \triangleright g$ and $\pi \triangleright \mathrm{id} = \pi$.
- For any $f : \mathcal{M}(R, S)$, $g : \mathcal{M}(S, T)$ and $\xi \Vdash^{o} T$, we have $(f; g) \triangleleft \xi = f \triangleleft (g \triangleleft \xi)$ and $\mathrm{id} \triangleleft \xi = \xi$.

Definition 1.2. Suppose that $\mathcal{M}$ has proofs. The glued category $\mathbb{G}(\mathcal{M})$ can be defined as follows:

- Its objects are triples (R, Π, Ξ) where R is an object of $\mathcal{M}$, Π is a set of proofs of R and Ξ is a set of counter-proofs of R.
- Morphisms $(R, \Pi, \Xi) \xrightarrow{f} (S, \Lambda, \Sigma)$ are morphisms $f : \mathcal{M}(R, S)$ such that:
  - for any $\pi \in \Pi$, $\pi \triangleright f \in \Lambda$ (i.e. f preserves proofs);
  - for any $\sigma \in \Sigma$, $f \triangleleft \sigma \in \Xi$ (i.e. f preserves counter-proofs).

Remark. In the following parts, proofs and counter-proofs will be morphisms from the host category $\mathcal{M}$ endowed with the usual categorical composition.

[3] The notation for proofs and counter-proofs is inherited both from realizability and games, as a player (p) proof is an opponent (o) counter-proof, and conversely.

Definition 1.3. Suppose that $\mathcal{M}$ features a given structure, such as a binary connective $R \odot S$ together with structural maps. Whenever the conditions are sufficient, we can lift up this structure from $\mathcal{M}$ to $\mathbb{G}(\mathcal{M})$, as:

$$(R, \Pi, \Xi) \odot (S, \Lambda, \Sigma) = (R \odot S, \Delta, \Omega)$$

With enough provisos, if we carefully choose Δ and Ω, the structural maps can also be lifted from $\mathcal{M}$ to $\mathbb{G}(\mathcal{M})$, giving rise to the same structure on the glued category, which in addition will be automatically compatible with the forgetful functor.

1.2 Orthogonalities

Once we have recovered $\mathbb{G}(\mathcal{M})$, which is quite huge, we may skim the junk out of it through orthogonality properties.

Definition 1.4. For any object R of $\mathcal{M}$, let $\perp_R$ be a relation between proofs and counter-proofs of type R. Let Π be a set of R-proofs and Ξ a set of R-counter-proofs. We define their respective orthogonal as follows:

$$\Pi^\bullet = \{\xi \Vdash^{o} R \mid \forall \pi \in \Pi, \pi \perp_R \xi\}$$
$$\Xi^\bullet = \{\pi \Vdash^{p} R \mid \forall \xi \in \Xi, \pi \perp_R \xi\}$$

We will call orthogonalities such families of relations.

Lemma 1.5. *As usual, $\perp$ defines a Galois connection, hence the following properties hold:*

- $A \subseteq A^{\bullet\bullet}$
- $A^{\bullet\bullet\bullet} = A^\bullet$
- $A \subseteq B$ *implies* $B^\bullet \subseteq A^\bullet$

Definition 1.6. Let $\perp$ be an orthonality, then $\mathbb{G}(\mathcal{M})$ gives rise to two important subcategories:

- The *slack* category $\mathbb{S}(\mathcal{M})$, which is the full subcategory of $\mathbb{G}(\mathcal{M})$ whose objects (R, Π, Ξ) are such that for any $\pi \in \Pi$ and for any $\xi \in \Xi$, $\pi \perp_R \xi$.
- The *tight* category $\mathbb{T}(\mathcal{M})$, which is the full subcategory of $\mathbb{G}(\mathcal{M})$ whose objects (R, Π, Ξ) are such that $\Pi = \Pi^{\bullet\bullet}$ and $\Xi = \Pi^\bullet$ (or dually, $\Xi = \Xi^{\bullet\bullet}$ and $\Pi = \Xi^\bullet$).

In particular, a set A such that $A = A^{\bullet\bullet}$ will be called closed.

Remark. As for the glued category, with enough conditions we can lift structure from $\mathcal{M}$ to either $\mathbb{S}(\mathcal{M})$ or $\mathbb{T}(\mathcal{M})$. This will be at heart of the next sections.

Lemma 1.7. $\mathbb{G}(\mathbf{C}) = \mathbb{S}(\mathbf{C})$ *for the full orthogonality.*

We define here some properties of morphisms that will be useful afterwards.

Definition 1.8. Suppose we have an orthogonality $\perp$ on $\mathcal{M}$. Let $f : \mathcal{M}(R, S)$, Π a set of R-proofs and Λ a set of S-counter-proofs.

- We say that f is *positive* w.r.t. Π and Λ if for any $\pi \in \Pi$ and any $\lambda \in \Lambda$,

  $$\pi \triangleright f \perp_S \lambda \text{ implies } \pi \perp_R f \triangleleft \lambda.$$

- We say that f is *negative* w.r.t. Π and Λ when it satisfies the reverse implication.
- If a morphism is a the same time positive and negative, we say it is *focussed*.

2 Enriched models

2.1 Motivations

Enriched models are at the intersection of various parts of logic and semantics.

In their most general case, they are a refinement of symmetric monoidal closed categories [12], and are also close to adjunction-based models of linear logic [1]. Hence, they are particularly well-fitted to study linear logic. Moreover, they naturally give rise to a *polarized* linear framework, and they are similar in expressivity to dialogue categories [13].

Enriched models are also related to Levy's call-by-push-value [11], and as such may be taken as models of programming languages featuring effects. This is related to the asymmetry of the pseudo-tensor $\downarrow A \otimes \underline{B}$ which naturally lifts to a premonoidal tensor [3].

In a totally different realm, this pseudo-tensor may be a way to give a typing discipline to classical realizability. A pseudo-tensor is typically the type of a stack, and the disbalance between values and computations is likely to parallel the disjoint universes of terms and stacks.

2.2 Definitions

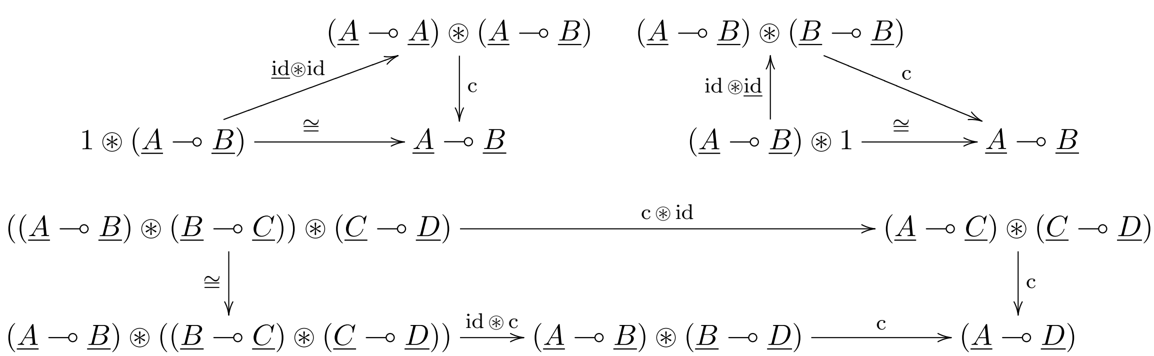
As its name implies, enriched models heavily rely on the notion of enriched categories, which can be seen as a way to internalize the homsets of a given category into another one.

Definition 2.1. Given a monoidal category $(\mathbf{V}, \circledast, 1)$, a $\mathbf{V}$-enriched category $\mathbf{C}$ is given as:

- a class of $\mathbf{C}$-objects $\underline{A}$, $\underline{B}$, $\underline{C}$...
- for any $\mathbf{C}$-objects $\underline{A}$ and $\underline{B}$, an object $\underline{A} \multimap \underline{B}$ of $\mathbf{V}$
- for any $\mathbf{C}$-objects $\underline{A}$, $\underline{B}$ and $\underline{C}$, two morphisms:

$$\underline{\mathrm{id}}_{\underline{A}} : \mathbf{V}(1, \underline{A} \multimap \underline{A})$$
$$\mathrm{c}_{\underline{A},\underline{B},\underline{C}} : \mathbf{V}((\underline{A} \multimap \underline{B}) \circledast (\underline{B} \multimap \underline{C}), \underline{A} \multimap \underline{C})$$

such that the following diagrams commute:



Lemma 2.2. *Any* $\mathbf{V}$*-enriched category* $\mathbf{C}$ *is a category on its own, with the same objects and* $\mathbf{C}(\underline{A}, \underline{B}) := \mathbf{V}(1, \underline{A} \multimap \underline{B})$*. Identity is the enriched identity*

$$\underline{\mathrm{id}}_{\underline{A}} : \mathbf{C}(\underline{A}, \underline{A})$$

and for any $f : \mathbf{C}(\underline{A}, \underline{B})$ *and* $g : \mathbf{C}(\underline{B}, \underline{C})$*, the* $\mathbf{C}$*-composite* $f \cdot g : \mathbf{C}(\underline{A}, \underline{C})$ *is*

$$f \cdot g : \; 1 \xrightarrow{\cong} 1 \circledast 1 \xrightarrow{f \circledast g} (\underline{A} \multimap \underline{B}) \circledast (\underline{B} \multimap \underline{C}) \xrightarrow{\mathrm{c}} \underline{A} \multimap \underline{C}$$

Remark. From now on, we will identify $\mathbf{V}(1, \underline{A} \multimap \underline{B})$ and $\mathbf{C}(\underline{A}, \underline{B})$.

Definition 2.3. Let $\mathbf{C}$ be a $\mathbf{V}$-enriched category.

We say that $\mathbf{C}$ has $\mathbf{V}$-tensors if for any $\mathbf{V}$-object A and $\mathbf{C}$-object $\underline{B}$ there exist a $\mathbf{C}$-object $\downarrow A \otimes \underline{B}$ and $\underline{C}$-natural bijections:

$$\varphi_{A,\underline{B},\underline{C}} : \mathbf{V}(A, \underline{B} \multimap \underline{C}) \cong \mathbf{C}(\downarrow A \otimes \underline{B}, \underline{C})$$

We say that $\mathbf{C}$ has $\mathbf{V}$-cotensors if for any $\mathbf{V}$-object A and $\mathbf{C}$-object $\underline{B}$ there exist a $\mathbf{C}$-object $\uparrow A^{\perp} \parr \underline{B}$ and $\underline{C}$-natural bijections:

$$\psi_{A,\underline{B},\underline{C}} : \mathbf{V}(A, \underline{C} \multimap \underline{B}) \cong \mathbf{C}(\underline{C}, \uparrow A^{\perp} \parr \underline{B})$$

Enriched naturality of φ and ψ amounts to the following equalities, for any $f : \mathbf{V}(A, \underline{B} \multimap \underline{C})$, $g : \mathbf{C}(\underline{C}, \underline{C'})$ and $h : \mathbf{C}(\underline{B'}, \underline{B})$:

$$\rho^{-1}; \varphi(f) \circledast g; \mathrm{c} = \varphi(\rho^{-1}; f \circledast g; \mathrm{c})$$

$$\lambda^{-1}; h \circledast \psi(f); \mathrm{c} = \psi(\lambda^{-1}; h \circledast f; \mathrm{c})$$

We will call any category with $\mathbf{V}$-enriched tensors and cotensors a $\mathbf{V}$-enriched model. $\mathbf{V}$ is the *value* category, while $\mathbf{C}$ is the *computation* category.

Remark. As the watchful reader may have noticed, the above notations are heavily inspired by polarized linear logic. Actually, it is not unwise to keep this in mind when heading towards the next part.

2.3 Zoology

As stated in the introductory part of this section, a lot of categorical structures from the literature are instances of enriched models.

Lemma 2.4. *Any (symmetric) monoidal closed category is a self-enriched model, with the obvious constructions.*

Definition 2.5. A LNL model [1] is a triple $(\mathbf{C}, \mathbf{V}, F \dashv G)$ where:

- $\mathbf{C}$ is a symmetric monoidal closed category.
- $\mathbf{V}$ is a cartesian category.
- There exist a monoidal adjunction: $\mathbf{V} \underset{G}{\overset{F}{\underset{\perp}{\rightleftarrows}}} \mathbf{C}$

Lemma 2.6. *Let* $(\mathbf{C}, \mathbf{V}, F \dashv G)$ *be a LNL model. Then it gives rise to a* $\mathbf{V}$*-enriched model with the following structure:*

$$\begin{aligned} \underline{A} \multimap \underline{B} &= G(\underline{A} \multimap \underline{B}) \\ \downarrow\!A \otimes \underline{B} &= FA \otimes \underline{B} \\ \uparrow\!A^{\perp} \parr \underline{B} &= FA \multimap \underline{B} \end{aligned}$$

Definition 2.7. A dialogue category is a symmetric monoidal category $\mathbf{C}$ together with a functor functor $\neg : \mathbf{C} \to \mathbf{C}^{op}$, such that there exists a natural isomorphism:

$$\mathbf{C}(A \otimes B, \neg C) \cong \mathbf{C}(A, \neg(B \otimes C))$$

Lemma 2.8. *Let* $(\mathbf{C}, \neg)$ *be a dialogue category. We can construct a* $\mathbf{C}$*-enriched model as follows:*

$$\begin{aligned} A \multimap B &= \neg(A \otimes \neg B) \\ \downarrow\!A \otimes B &= A \otimes B \\ \uparrow\!A^{\perp} \parr B &= \neg(A \otimes \neg B) \end{aligned}$$

Lemma 2.9. *Let* $\mathbf{C}$ *be a* $\mathbf{V}$*-enriched model, and let* $\underline{1}$ *and* $\underline{\perp}$ *be two* $\mathbf{C}$*-objects. Then* $(\mathbf{V}, \neg)$ *is a dialogue category, where:*

$$\neg A = (\downarrow\!A \otimes \underline{1}) \multimap \underline{\perp} \cong \underline{1} \multimap (\uparrow\!A^{\perp} \parr \underline{\perp})$$

The previous lemmas give a better explanation of what an enriched model is: this is essentially a dialogue category with explicit (and eventually many) entry and return types. Another difference lies in the fact that dialogue categories are depolarized: the negation functor is internal to the category, while enriched categories discriminate between depolarized ($\underline{A} \multimap \underline{B}$) and negative ($\uparrow\!A^{\perp} \parr \underline{B}$) function spaces.

2.4 Generic constructions

We define here some useful constructions on morphisms that will be needed later on.

Definition 2.10. Let $\mathbf{V}$ be a monoidal category, $u : \mathbf{V}(1, R)$, $v : \mathbf{V}(1, S)$ and $h : \mathbf{V}(R \circledast S, T)$. The cutting of h along u is the morphism:

$$h[u] : \; S \xrightarrow{\cong} 1 \circledast S \xrightarrow{u \circledast S} R \circledast S \xrightarrow{h} T$$

We can symmetrically obtain $h[v] : \mathbf{V}(R, T)$.

Let $\mathbf{C}$ be a $\mathbf{V}$-enriched model. The enriched structure defines a lot of enriched functors on $\mathbf{C}$ and $\mathbf{V}$. An enriched functor can be simply seen as a usual functor on the underlying plain categories.

Definition 2.11. For any $f : \mathbf{C}(\underline{A}, \underline{B})$ and $g : \mathbf{C}(\underline{C}, \underline{D})$, we can define the functorial pre and post-composition:

$$f \multimap \underline{C} : \; \underline{B} \multimap \underline{C} \xrightarrow{\cong} 1 \circledast (\underline{B} \multimap \underline{C}) \xrightarrow{f \circledast (\underline{B} \multimap \underline{C})} (\underline{A} \multimap \underline{B}) \circledast (\underline{B} \multimap \underline{C}) \xrightarrow{\mathrm{c}} \underline{A} \multimap \underline{C}$$

$$\underline{B} \multimap g :\ \underline{B} \multimap \underline{C} \xrightarrow{\cong} (\underline{B} \multimap \underline{C}) \circledast 1 \xrightarrow{(\underline{B}\multimap\underline{C})\circledast g} (\underline{B} \multimap \underline{C}) \circledast (\underline{C} \multimap \underline{D}) \xrightarrow{\mathrm{c}} \underline{B} \multimap \underline{D}$$

Thanks to associativity of the enriched composition, $f \multimap g : \mathbf{V}(\underline{B} \multimap \underline{C}, \underline{A} \multimap \underline{D})$ is well defined:

$$f \multimap g = (f \multimap \underline{C}) \cdot (\underline{A} \multimap g) = (\underline{B} \multimap g) \cdot (f \multimap \underline{D})$$

Definition 2.12. The enriched tensor also defines two enriched functors. For any $f : \mathbf{V}(A, B)$, let $\downarrow f \otimes \underline{C} : \mathbf{C}(\downarrow A \otimes \underline{C}, \downarrow B \otimes \underline{C})$ be:

$$\varphi^{-1}(\downarrow f \otimes \underline{C}) :\ A \xrightarrow{f} B \xrightarrow{\varphi^{-1}(\underline{\mathrm{id}})} \underline{C} \multimap \downarrow B \otimes \underline{C}$$

and for any $g : \mathbf{C}(\underline{A}, \underline{B})$ let $\downarrow C \otimes g : \mathbf{C}(\downarrow C \otimes \underline{A}, \downarrow C \otimes \underline{B})$ be:

$$\varphi^{-1}(\downarrow C \otimes g) :\ C \xrightarrow{\cong} 1 \circledast C \xrightarrow{g \circledast \varphi^{-1}(\underline{\mathrm{id}})} (\underline{A} \multimap \underline{B}) \circledast (\underline{B} \multimap \downarrow C \otimes \underline{B}) \xrightarrow{\mathrm{c}} \underline{A} \multimap \downarrow C \otimes \underline{B}$$

We can define $\uparrow f^{\perp} \parr \underline{C} : \mathbf{C}(\uparrow B^{\perp} \parr \underline{C}, \uparrow A^{\perp} \parr \underline{C})$ and $\uparrow C^{\perp} \parr g : \mathbf{C}(\uparrow C^{\perp} \parr \underline{A}, \uparrow C^{\perp} \parr \underline{B})$ the very same way[4].

Definition 2.13. For any $u : \mathbf{V}(1, A)$, let $\downarrow u : \mathbf{C}(\underline{B}, \downarrow A \otimes \underline{B})$ be:

$$\downarrow u :\ 1 \xrightarrow{u} A \xrightarrow{\varphi^{-1}(\underline{\mathrm{id}})} \underline{B} \multimap \downarrow A \otimes \underline{B}$$

and $\uparrow u : \mathbf{C}(\uparrow A^{\perp} \parr \underline{B}, \underline{B})$ be:

$$\uparrow u :\ 1 \xrightarrow{u} A \xrightarrow{\psi^{-1}(\underline{\mathrm{id}})} \uparrow A^{\perp} \parr \underline{B} \multimap \underline{B}$$

3 Double-glueing of enriched models

Double-glueing over enriched models is a little bit tedious, as we now need to take care of two categories. This leads to a tedious duplication of most of the aforementioned definitions.

In the following section, we will consider $\mathbf{V}$ a symmetric monoidal category, and $\mathbf{C}$ a $\mathbf{V}$-enriched model. Let $\underline{1}$ and $\underline{\perp}$ two distinguished $\mathbf{C}$-objects.

3.1 Slack categories

Definition 3.1 ($\mathbf{V}$-orthogonality)**.** A $\mathbf{V}$-orthogonality $\perp^{\mathbf{V}}$ is a family of relations

$$\perp_R^{\mathbf{V}} \subseteq \mathbf{V}(1, R) \times \mathbf{V}(R, \underline{1} \multimap \underline{\perp})$$

such that the following holds:

Identity: for any $u : \mathbf{V}(1, R)$ and $x : \mathbf{V}(R, \underline{1} \multimap \underline{\perp})$,

$$u \perp_R x \text{ implies } \mathrm{id}_1 \perp_1 u; x$$

[4] Yet the first functor is contravariant.

Tensor: for any $u : \mathbf{V}(1,R)$, $v : \mathbf{V}(1,S)$ and $z : \mathbf{V}(R \circledast S, \underline{1} \multimap \underline{\bot})$,

$$u \perp_R z[v] \text{ and } v \perp_S z[u] \text{ imply } \rho^{-1}; u \circledast v \perp_{R \circledast S} z$$

For any $U \subseteq \mathbf{V}(1,R)$ and $X \subseteq \mathbf{V}(R, \underline{1} \multimap \underline{\bot})$, we will note $U^\bullet$ and $X^\bullet$ the orthogonal sets, as before.

Definition 3.2. If $\perp^{\mathbf{V}}$ is a $\mathbf{V}$-orthogonality, let us define the slack category $\mathbb{S}(\mathbf{V})$ as follows:

- Objects are triples (R, U, X) where $R \in \mathbf{V}$, $U \subseteq \mathbf{V}(1,R)$ and $X \subseteq \mathbf{V}(R, \underline{1} \multimap \underline{\bot})$, where $U \perp^{\mathbf{V}} X$
- Morphisms between (R, U, X) and (S, V, Y) are morphisms $f : \mathbf{V}(R,S)$ such that:

  Forward preservation: for all $u \in U$, $u; f \in V$

  Reverse preservation: for all $y \in Y$, $f; y \in X$

Theorem 3.3. $\mathbb{S}(\mathbf{C})$ *is a symmetric monoidal category, where:*

- $1 = (1, \{\mathrm{id}_1\}, \{\mathrm{id}_1\}^\bullet)$
- $(R, U, X) \circledast (S, V, Y) = (R \circledast S, U \circledast V, Z)$ *with:*

$$U \circledast V = \{\rho^{-1}; u \circledast v \mid u \in U, v \in V\}$$

$$Z = \{z : \mathbf{V}(R \circledast S, \underline{1} \multimap \underline{\bot}) \mid \forall u \in U, z[u] \in Y \wedge \forall v \in V, z[v] \in X\}$$

Definition 3.4 ($\mathbf{C}$-orthogonality)**.** Let $\perp^{\mathbf{V}}$ be a $\mathbf{V}$-orthogonality. An $\mathbf{C}$-orthogonality $\perp^{\mathbf{C}}$ is a family of relations

$$\perp^{\mathbf{C}}_{\underline{R}} \subseteq \mathbf{C}(\underline{1}, \underline{R}) \times \mathbf{C}(\underline{R}, \underline{\bot})$$

such that the following holds:

Identity: for any $u : \mathbf{C}(\underline{1}, \underline{R})$ and $x : \mathbf{C}(\underline{R}, \underline{\bot})$,

$$u \perp^{\mathbf{C}}_{\underline{R}} x \text{ implies } \mathrm{id}_1 \perp^{\mathbf{V}}_1 u \cdot x$$

Arrow: for any $w : \mathbf{C}(\underline{R}, \underline{S})$, $u : \mathbf{C}(\underline{1}, \underline{R})$ and $y : \mathbf{C}(\underline{S}, \underline{\bot})$,

$$u \cdot w \perp^{\mathbf{C}}_{\underline{S}} y \text{ and } u \perp^{\mathbf{C}}_{\underline{R}} w \cdot y \text{ imply } w \perp^{\mathbf{V}}_{\underline{R} \multimap \underline{S}} u \multimap y$$

Tensor: for any $u : \mathbf{V}(1,R)$, $v : \mathbf{C}(\underline{1}, \underline{S})$ and $z : \mathbf{C}(\downarrow R \otimes \underline{S}, \underline{\bot})$,

$$u \perp^{\mathbf{V}}_{R} \downarrow A \otimes v \cdot z \text{ and } v \perp^{\mathbf{C}}_{\underline{S}} \downarrow u \cdot z \text{ imply } v \cdot \downarrow u \perp^{\mathbf{C}}_{\downarrow R \otimes \underline{S}} z$$

Cotensor: for any $u : \mathbf{V}(1,R)$, $y : \mathbf{C}(\underline{S}, \underline{\bot})$ and $w : \mathbf{C}(\underline{1}, \uparrow R^\perp \parr \underline{S})$,

$$u \perp^{\mathbf{V}}_{R} w \cdot \uparrow A^\perp \parr y \text{ and } w \cdot \uparrow u \perp^{\mathbf{C}}_{\underline{S}} y \text{ imply } w \perp^{\mathbf{C}}_{\uparrow R^\perp \parr \underline{S}} \uparrow u \cdot y$$

In order to visually discriminate orthogonal sets for $\perp^{\mathbf{V}}$ and $\perp^{\mathbf{C}}$, we will note the orthogonal of a set A w.r.t. $\perp^{\mathbf{C}}$ as A°.

Definition 3.5. If $\perp^{\mathbf{C}}$ is a $\mathbf{C}$-orthogonality, we define the slack category $\mathbb{S}(\mathbf{C})$ as follows:

- Objects are triples $(\underline{R}, \underline{U}, \underline{X})$ where $\underline{R} \in \mathbf{C}$, $\underline{U} \subseteq \mathbf{C}(\underline{1}, \underline{R})$ and $\underline{X} \subseteq \mathbf{C}(\underline{R}, \underline{\perp})$, where $\underline{U} \perp^{\mathbf{C}} \underline{X}$
- Morphisms between $(\underline{R}, \underline{U}, \underline{X})$ and $(\underline{S}, \underline{V}, \underline{Y})$ are morphisms $f : \mathbf{C}(\underline{R}, \underline{S})$ such that:

  Forward preservation: for all $u \in \underline{U}$, $u \cdot f \in \underline{V}$

  Reverse preservation: for all $y \in \underline{Y}$, $f \cdot y \in \underline{X}$

Theorem 3.6. *The category $\mathbb{S}(\mathbf{C})$ is an $\mathbb{S}(\mathbf{V})$-enriched model, with:*

- $(\underline{R}, \underline{U}, \underline{X}) \multimap (\underline{S}, \underline{V}, \underline{Y}) = (\underline{R} \multimap \underline{S}, W, U \multimap Y)$ *where*

$$W = \{w : \mathbf{C}(\underline{R}, \underline{S}) \mid \forall u \in \underline{U}, u \cdot w \in \underline{V} \wedge \forall y \in \underline{Y}, w \cdot y \in \underline{X}\}$$

$$U \multimap Y = \{u \multimap y \mid u \in \underline{U}, y \in \underline{Y}\}$$

- $\downarrow(R, U, X) \otimes (\underline{S}, \underline{V}, \underline{Y}) = (\downarrow R \otimes \underline{S}, \downarrow U \otimes \underline{V}, \underline{Z})$ *where*

$$\downarrow U \otimes \underline{V} = \{v \cdot \downarrow u \mid u \in U, v \in \underline{V}\}$$

$$\underline{Z} = \{z : \mathbf{C}(\downarrow R \otimes \underline{S}, \underline{\perp}) \mid \forall u \in U, \downarrow u \cdot z \in \underline{Y} \wedge \forall v \in \underline{V}, \downarrow R \otimes v \cdot z \in X\}$$

- $\uparrow(R, U, X)^{\perp} \parr (\underline{S}, \underline{V}, \underline{Y}) = (\uparrow R^{\perp} \parr \underline{S}, \underline{W}, \uparrow U^{\perp} \parr \underline{Y})$ *where*

$$\underline{W} = \{w : \mathbf{C}(\underline{1}, \uparrow R^{\perp} \parr \underline{S}) \mid \forall u \in U, w \cdot \downarrow u \in \underline{V} \wedge \forall y \in \underline{Y}, w \cdot \downarrow R \otimes y \in X\}$$

$$\uparrow U^{\perp} \parr \underline{Y} = \{\uparrow u \cdot y \mid u \in U, y \in \underline{Y}\}$$

3.2 Tight categories

Let $\mathbf{C}$ be a $\mathbf{V}$-enriched model equipped with orthogonalities.

Definition 3.7 (Tight category)**.** The tight category $\mathbb{T}(\mathbf{V})$ is defined as the full subcategory of $\mathbb{S}(\mathbf{V})$ whose objects are of the form (R, U, X) where $U^{\bullet\bullet} = U$ and $X = U^{\bullet}$. We call such sets closed.

Likewise, the tight category $\mathbb{T}(\mathbf{C})$ is defined as the full subcategory of $\mathbb{S}(\mathbf{C})$ whose objects are of the form $(\underline{R}, \underline{U}, \underline{X})$ where $\underline{U}^{\circ\circ} = \underline{U}$ and $\underline{X} = \underline{U}^{\circ}$.

Remark. Any object $(R, U, X) \in \mathbb{T}(\mathbf{V})$ it is entirely described by only one of the two sets U and X. Therefore, we will define them as (R, U) where $U^{\bullet\bullet} = U$ (and hence $X = U^{\bullet}$).

Definition 3.8 (Precise orthogonality)**.** We say that $\perp^{\mathbf{V}}$ is precise whenever the following holds:

Identity: for any $u : \mathbf{V}(1, R)$, $x : \mathbf{V}(R, \underline{1} \multimap \underline{\perp})$,

$$u \perp_R x \text{ implies } \mathrm{id}_1 \perp_1 u; x$$

Tensor: for any $u : \mathbf{V}(1, R)$, $v : \mathbf{V}(1, S)$ and $z : \mathbf{V}(R \circledast S, \underline{1} \multimap \underline{\bot})$,

$$u \perp_R z[v] \text{ and } v \perp_S z[u] \text{ if and only if } \rho^{-1}; u \circledast v \perp_{R \circledast S} z$$

Definition 3.9 (**V**-stability)**.** For any $U \subseteq \mathbf{V}(1, R)$ and $Y \subseteq \mathbf{V}(S, \underline{1} \multimap \underline{\bot})$, and let us pose:

$$\mathbf{V}(U, Y) = \{f : \mathbf{V}(R, S) \mid \forall u \in U, u; f \in Y^\bullet \wedge \forall y \in Y, f; y \in U^\bullet\}$$

We say that $\perp^{\mathbf{V}}$ is stable whenever it is precise and the following equalities hold for any sets $U \subseteq \mathbf{V}(1, R)$, $V \subseteq \mathbf{V}(1, S)$ and $Y \subseteq \mathbf{V}(S, \underline{1} \multimap \underline{\bot})$:

$$(U^{\bullet\bullet} \circledast V^{\bullet\bullet})^\bullet = (U \circledast V^{\bullet\bullet})^\bullet = (U^{\bullet\bullet} \circledast V)^\bullet$$

$$\mathbf{V}(U^{\bullet\bullet}, Y^{\bullet\bullet}) = \mathbf{V}(U, Y^{\bullet\bullet}) = \mathbf{V}(U^{\bullet\bullet}, Y)$$

Lemma 3.10. *If $\perp^{\mathbf{V}}$ is a precise orthogonality, and A and B are closed, then the second component of $A \circledast B$ in $\mathbb{S}(\mathbf{V})$ is already closed.*

Theorem 3.11. *Suppose that $\perp^{\mathbf{V}}$ is stable, and that the SMC canonical isomorphisms are focussed. Then $\mathbb{T}(\mathbf{V})$ is a symmetric monoidal category with the following structure:*

- $1 = (1, \{\mathrm{id}_1\}^{\bullet\bullet})$
- $(R, U) \circledast (S, V) = (R \circledast S, (U \circledast V)^{\bullet\bullet})$

Definition 3.12. We say that $\perp^{\mathbf{C}}$ is precise when:

Identity: for any $u : \mathbf{C}(\underline{1}, \underline{R})$ and $x : \mathbf{C}(\underline{R}, \underline{\bot})$,

$$u \perp^{\mathbf{C}}_{\underline{R}} x \text{ implies } \mathrm{id}_1 \perp^{\mathbf{V}}_1 u \cdot x$$

Arrow: for any $w : \mathbf{C}(\underline{R}, \underline{S})$, $u : \mathbf{C}(\underline{1}, \underline{R})$ and $y : \mathbf{C}(\underline{S}, \underline{\bot})$,

$$u \cdot w \perp^{\mathbf{C}}_{\underline{S}} y \text{ and } u \perp^{\mathbf{C}}_{\underline{R}} w \cdot y \text{ if and only if } w \perp^{\mathbf{V}}_{\underline{R} \multimap \underline{S}} u \multimap y$$

Tensor: for any $u : \mathbf{V}(1, R)$, $v : \mathbf{C}(\underline{1}, \underline{S})$ and $z : \mathbf{C}({\downarrow}R \otimes \underline{S}, \underline{\bot})$,

$$u \perp^{\mathbf{V}}_R {\downarrow}A \otimes v \cdot z \text{ and } v \perp^{\mathbf{C}}_{\underline{S}} {\downarrow}u \cdot z \text{ if and only if } v \cdot {\downarrow}u \perp^{\mathbf{C}}_{{\downarrow}R \otimes \underline{S}} z$$

Cotensor: for any $u : \mathbf{V}(1, R)$, $y : \mathbf{C}(\underline{S}, \underline{\bot})$ and $w : \mathbf{C}(\underline{1}, {\uparrow}R^\perp \parr \underline{S})$,

$$u \perp^{\mathbf{V}}_R w \cdot {\uparrow}A^\perp \parr y \text{ and } w \cdot {\uparrow}u \perp^{\mathbf{C}}_{\underline{S}} y \text{ if and only if } w \perp^{\mathbf{C}}_{{\uparrow}R^\perp \parr \underline{S}} {\uparrow}u \cdot y$$

Lemma 3.13. *If $\perp^{\mathbf{C}}$ is precise, and A, $\underline{B}$ and $\underline{C}$ are closed (for their respective orthogonalities), then the first component of ${\uparrow}A^\perp \parr \underline{B}$ and $\underline{B} \multimap \underline{C}$ and the second component of ${\downarrow}A \otimes \underline{B}$ are already closed in their respective slack category.*

Definition 3.14 (**C**-stability)**.** We say that $\perp^{\mathbf{C}}$ is stable when it is precise and the following holds:

Identity left: for any $u \in \mathbf{C}(\underline{1}, \underline{R})$, $x \in \mathbf{C}(\underline{R}, \underline{\bot})$ and $v \in \{\mathrm{id}_1\}^{\bullet\bullet}$,

$$u \perp^{\mathbf{C}}_{\underline{R}} x \text{ implies } v; u \perp^{\mathbf{C}}_{\underline{R}} x$$

Identity right: for any $u \in \mathbf{C}(\underline{1}, \underline{R})$, $x \in \mathbf{C}(\underline{R}, \underline{\perp})$ and $v \in \{\mathrm{id}_1\}^{\bullet\bullet}$,

$$u \perp^{\mathbf{C}}_{\underline{R}} x \text{ implies } u \perp^{\mathbf{C}}_{\underline{R}} v; x$$

Arrow: for any $\underline{U} \subseteq \mathbf{C}(\underline{1}, \underline{R})$ and $\underline{Y} \subseteq \mathbf{C}(\underline{S}, \underline{\perp})$

$$(\underline{U}^{\circ\circ} \multimap \underline{Y}^{\circ\circ})^{\bullet} = (\underline{U} \multimap \underline{Y}^{\circ\circ})^{\bullet} = (\underline{U}^{\circ\circ} \multimap \underline{Y})^{\bullet}$$

Theorem 3.15. *If $\perp^{\mathbf{V}}$ and $\perp^{\mathbf{C}}$ are stable, then $\mathbb{T}(\mathbf{C})$ is a $\mathbb{T}(\mathbf{V})$-enriched model with the following structure:*

- $(\underline{R}, \underline{U}) \multimap (\underline{S}, \underline{V}) = (\underline{R} \multimap \underline{S}, (\underline{U} \multimap \underline{V}^{\circ})^{\bullet})$
- $\downarrow(R, U) \otimes (\underline{S}, \underline{V}) = (\downarrow R \otimes \underline{S}, (\downarrow U \otimes \underline{V})^{\circ\circ})$
- $\uparrow(R, U)^{\perp} \parr (\underline{S}, \underline{V}) = (\uparrow R^{\perp} \parr \underline{S}, (\uparrow U^{\perp} \parr \underline{V}^{\circ})^{\circ})$

3.3 Focussed orthogonalities

A special case of orthogonalities are the so-called focussed orthogonalities. Such orthogonalities are similar to those found in classical realizability and are particularly nice candidates to perform double-glueing.

Definition 3.16. A pair of orthogonalities $(\perp^{\mathbf{V}}, \perp^{\mathbf{C}})$ is focussed whenever there exists a pole $\perp\!\!\!\perp \subseteq \mathbf{V}(1, \underline{1} \multimap \underline{\perp}) = \mathbf{C}(\underline{1}, \underline{\perp})$ such that:

- for any $u : \mathbf{V}(1, R)$ and $x : \mathbf{V}(R, \underline{1} \multimap \underline{\perp})$, $u \perp^{\mathbf{V}} x$ iff $u; x \in \perp\!\!\!\perp$
- for any $u : \mathbf{C}(\underline{1}, \underline{R})$ and $x : \mathbf{C}(\underline{R}, \underline{\perp})$, $u \perp^{\mathbf{C}} x$ iff $u \cdot x \in \perp\!\!\!\perp$

Lemma 3.17. *Focussed orthogonalities are automatically compliant with any of the requirements mentionned in the previous sections, i.e. they are precise and stable orthogonalities.*

4 Application to SMCCs

As stated in lemma 2.4, monoidal closed categories are a special case of enriched models where $\mathbf{V} = \mathbf{C}$. As such, the previous results on double-glueing still hold, with various simplifications triggered by this identification. We will rephrase them in this section for the sake of readability.

We recall that the following collapses occur when considering an SMCC:

$$\downarrow A \otimes B = A \circledast B$$
$$\uparrow A^{\perp} \parr B = A \multimap B$$

We also identitify $\underline{1} = 1$, hence $\underline{1} \multimap \underline{\perp} \cong \underline{\perp}$. As linear logic is commutative, we will only consider *symmetric* monoidal closed categories.

In the following section, let $(\mathbf{C}, \otimes, 1)$ be a SMCC, and $\perp$ a distinguished return object. We will implicitely assume that $\perp$ is the dualizing object whenever $\mathbf{C}$ is $*$-autonomous.

4.1 Slack categories

Definition 4.1. Let $\perp_R \subseteq \mathbf{C}(1, R) \times \mathbf{C}(R, \perp)$ a family of relations. We say $\perp$ is an orthogonality if the following holds:

Identity: for any $u : \mathbf{C}(1, R)$ and $x : \mathbf{C}(R, \perp)$,

$$u \perp_R x \text{ implies } \mathrm{id}_1 \perp_1 u \cdot x$$

Tensor: for any $u : \mathbf{C}(1, R)$, $v : \mathbf{C}(1, S)$ and $z : \mathbf{C}(R \otimes S, \perp)$,

$$u \perp_R z[v] \text{ and } v \perp_S z[u] \text{ imply } \rho^{-1}; u \otimes v \perp_{R\otimes S} z$$

Arrow: for any $w : \mathbf{C}(R, S)$, $u : \mathbf{C}(1, R)$ and $y : \mathbf{C}(S, \perp)$,

$$u; w \perp_S y \text{ and } u \perp_R w; y \text{ imply } w \perp_{R\multimap S} u \multimap y$$

Dual: (when $\mathbf{C}$ is $*$-autonomous) for any $u : \mathbf{C}(1, R)$ and $x : \mathbf{C}(R, \perp)$,

$$u \perp_R x \text{ imply } x^* \perp_{R^*} u^*$$

Remark. Any orthogonality in an SMCC is in particular an orthogonality on the induced enriched model.

Definition 4.2. We define the slack category $\mathbb{S}(\mathbf{C})$ as in definition 1.6, where an R-proof is a morphism $u : \mathbf{C}(1, R)$ and an R-counter-proof is a morphism $x : \mathbf{C}(R, \perp)$. Hence objects are triples $A = (R, U \subseteq \mathbf{C}(1, R), X \subseteq \mathbf{C}(R, \perp))$, and morphisms must satisfy the usual forward and reverse composition compatibility. We note $u \Vdash^p A := u \in U$ and $x \Vdash^o A := x \in X$.

Lemma 4.3. *$\mathbb{S}(\mathbf{C})$ is a SMCC, with the following inductive structure:*

$$\frac{}{\mathrm{id}_1 \Vdash^p 1} \qquad \frac{\mathrm{id}_1 \perp \chi}{\chi \Vdash^o 1}$$

$$\frac{u_1 \Vdash^p A_1 \qquad u_2 \Vdash^p A_2}{u_1 \otimes u_2 \Vdash^p A_1 \otimes A_2} \qquad \frac{\forall u_i \Vdash^p A_i,\ z[u_i] \Vdash^o A_j}{z \Vdash^o A_1 \otimes A_2}$$

$$\frac{\forall u \Vdash^p A,\ u; w \Vdash^p B \qquad \forall y \Vdash^o B,\ w; y \Vdash^o A}{w \Vdash^p A \multimap B} \qquad \frac{u \Vdash^p A \qquad y \Vdash^o B}{u \multimap y \Vdash^o A \multimap B}$$

If moreover $\mathbf{C}$ is $*$-autonomous, then so is $\mathbb{S}(\mathbf{C})$, with:

$$\frac{u^* \Vdash^o A^*}{u \Vdash^p A} \qquad \frac{x^* \Vdash^p A^*}{x \Vdash^o A}$$

Remark. This presentation is more convenient to understand the double-glueing construction in light of realizability.

Indeed, a tensor is a (positive) linear conjunction, hence its introduction rule is well-behaved w.r.t. proofs: a proof of a tensor is a pair of proofs. Conversely, there is no such thing as an elimination rule for the tensor, and counter-proofs must be modelled through a universal property.

In a dual way, the (negative) linear implication is well-behaved w.r.t. counter-proofs, and exhibits the very same problem as the tensor regarding proofs.

In addition to the multiplicative fragment of linear logic, we can show that additives lift flawlessly from the base category to the slack category.

Lemma 4.4. *Suppose* **C** *has products (resp. coproducts) and that the projections (resp. injections) are positive (resp. negative), then so has* $\mathbb{S}(\mathbf{C})$*, with the following inductive structure:*

$$\frac{}{\top_1 \Vdash^{p} \top} \qquad \frac{}{0_\perp \Vdash^{o} 0}$$

$$\frac{u_1 \Vdash^{p} A_1 \qquad u_2 \Vdash^{p} A_2}{\langle u_1 \mid u_2\rangle \Vdash^{p} A_1 \,\&\, A_2} \qquad \frac{x_i \Vdash^{o} A_i}{\pi_i ; x_i \Vdash^{o} A_1 \,\&\, A_2}$$

$$\frac{u_i \Vdash^{p} A_i}{u_i ; \iota_i \Vdash^{p} A_1 \oplus A_2} \qquad \frac{x_1 \Vdash^{o} A_1 \qquad x_2 \Vdash^{o} A_2}{[x_1 \mid x_2] \Vdash^{o} A_1 \oplus A_2}$$

Remark. Additives are much more well-behaved with respect to the usual (intuitionistic) BHK interpretation and lack the asymmetrical properties of multiplicatives.

4.2 Polarization

Double-glueing in the SMCC case stresses out the polarization properties of linear logic [10]. Theses features can be traced back to lemma 3.13.

Definition 4.5. Similarly to the enriched case, we say that $\perp$ is precise whenever the tensor and arrow implications from definition 4.1 are equivalences.

Definition 4.6 (Polarized glueing)**.** Let $\mathbb{S}^+(\mathbf{C})$, the positive glueing of **C**, be the full subcategory of $\mathbb{S}(\mathbf{C})$ whose objects are of the form $(R, U, U^\bullet)$. We call such objects positive.

Dually, let $\mathbb{S}^-(\mathbf{C})$, the negative glueing of **C**, be the full subcategory of $\mathbb{S}(\mathbf{C})$ whose objects are of the form $(R, X^\bullet, X)$, which we call negative objects.

Lemma 4.7. *Suppose* $\perp$ *is a precise orthogonality. Then:*

- $\mathbb{S}^+(\mathbf{C})$ *is closed under* n*-ary tensor;*
- *for any* $P \in \mathbb{S}^+(\mathbf{C})$ *and* $N \in \mathbb{S}^-(\mathbf{C})$*, then* $P \multimap N \in \mathbb{S}^-(\mathbf{C})$*.*

Suppose in additition that **C** *is* $*$*-autonomous, then:*

- $\mathbb{S}^-(\mathbf{C})$ *is closed under* n*-ary par.*
- $N \in \mathbb{S}^-(\mathbf{C})$ *iff* $N^* \in \mathbb{S}^+(\mathbf{C})$*.*

Lemma 4.8. *Suppose that* **C** *has products (resp. coproducts) and that the projections (resp. injections) are focussed. Then* $\mathbb{S}^-(\mathbf{C})$ *(resp.* $\mathbb{S}^+(\mathbf{C})$*) is closed under* n*-ary products (resp. coproducts).*

These results mimic the usual properties of polarized linear logic, which discriminates between positive ($\otimes$, $\oplus$, 1, 0) and negative ($\parr$, &, $\perp$, $\top$) connectives.

Definition 4.9 (Polarized morphisms)**.** Let $\mathbb{S}^\downarrow(\mathbf{C})$ the subcategory of $\mathbb{S}(\mathbf{C})$ restricted to relative positive morphisms, i.e. $(R, U, X) \xrightarrow{f} (S, V, Y)$ is a morphism in $\mathbb{S}^\downarrow(\mathbf{C})$ if f is a positive morphism w.r.t. U and $V^\bullet$. In a dual way, $\mathbb{S}^\uparrow(\mathbf{C})$ is the subcategory of $\mathbb{S}(\mathbf{C})$ restricted to relative negative morphisms, i.e. $(R, U, X) \xrightarrow{f} (S, V, Y)$ is a negative morphism w.r.t. $X^\bullet$ and Y.

Remark. If every morphism of $\mathbf{C}$ is positive (resp. negative), $\mathbb{S}^{\downarrow}(\mathbf{C}) = \mathbb{S}(\mathbf{C})$ (resp. $\mathbb{S}^{\uparrow}(\mathbf{C}) = \mathbb{S}(\mathbf{C})$).

Lemma 4.10. *Let us define the following operations on objects of* $\mathbb{S}(\mathbf{C})$*:*

$$\downarrow(R, U, X) = (R, U, U^{\bullet})$$
$$\uparrow(R, U, X) = (R, X^{\bullet}, X)$$

Then $\downarrow$ *can be lifted to a functor* $\downarrow : \mathbb{S}^{\downarrow}(\mathbf{C}) \to \mathbb{S}^{+}(\mathbf{C})$ *while* $\uparrow$ *can be lifted to a functor* $\uparrow : \mathbb{S}^{\uparrow}(\mathbf{C}) \to \mathbb{S}^{-}(\mathbf{C})$*. Both admit the forgetful functor as an adjoint, but in a symmetric fashion:*

$$\mathbb{S}^{+}(\mathbf{C})(A, \downarrow B) \cong \mathbb{S}^{\downarrow}(\mathbf{C})(A, B)$$
$$\mathbb{S}^{-}(\mathbf{C})(\uparrow A, B) \cong \mathbb{S}^{\uparrow}(\mathbf{C})(A, B)$$

These adjunction results are close to those that can be found in an unpublished work by Melliès and Selinger on polar categories [15].

4.3 Tight categories

The tight construction is the most prominent instance of double-glueing in literature. While slack categories do appear here and there in some papers, they are by far outnumbered by the famous models of linear logic which are almost all built over a tight category.

This oddity can be easily explained when considering polarization issues: plain linear logic is a depolarized logic, while tight categories are, in a way, a depolarized version of slack categories, whence the correspondance LL/tight.

Definition 4.11. The tight category $\mathbb{T}(\mathbf{C})$ is the full subcategory of $\mathbb{S}(\mathbf{C})$ whose objects are of the form $(R, U^{\bullet\bullet}, U^{\bullet})$ for some $U \subseteq \mathbf{C}(1, R)$, or equivalently, $(R, X^{\bullet}, X^{\bullet\bullet})$, for some $X \subseteq \mathbf{C}(R, \perp)$.

Remark. We can rephrase the previous definition: the tight category is the full subcategory whose objects are both positive and negative.

In order to lift the multiplicative structure from $\mathbf{C}$ to $\mathbb{T}(\mathbf{C})$, we need to ensure that the corresponding tensor and arrow are indeed objects of $\mathbb{T}(\mathbf{C})$, whence the need for closing their structure from $\mathbb{S}(\mathbf{C})$.

Lemma 4.12. *Suppose* $\perp$ *is precise, and let* A *and* B *two objects of* $\mathbb{T}(\mathbf{C})$*. Then the following are objects of* $\mathbb{T}(\mathbf{C})$*:*

$$1_{\mathbb{T}(\mathbf{C})} = \uparrow(1_{\mathbb{S}(\mathbf{C})})$$
$$A \otimes_{\mathbb{T}(\mathbf{C})} B = \uparrow(A \otimes_{\mathbb{S}(\mathbf{C})} B)$$
$$A \multimap_{\mathbb{T}(\mathbf{C})} B = \downarrow(A \multimap_{\mathbb{S}(\mathbf{C})} B)$$

The original article [9] defined an awfully adhoc notion[5] that ensured the canonical monoidal morphisms existed in $\mathbb{T}(\mathbf{C})$, but we shall not discuss it there and shall directly head towards the natural property.

[5] Namely, self-stability.

Definition 4.13. We say that $\perp$ is *stable* whenever the following equalities hold for any $U \subseteq \mathbf{C}(1,R)$, $V \subseteq \mathbf{C}(1,S)$ and $Y \subseteq \mathbf{C}(S,\perp)$:

$$(U^{\bullet\bullet} \otimes V^{\bullet\bullet})^{\bullet} = (U \otimes V^{\bullet\bullet})^{\bullet} = (U^{\bullet\bullet} \otimes V)^{\bullet}$$
$$(U^{\bullet\bullet} \multimap Y^{\bullet\bullet})^{\bullet} = (U \multimap V^{\bullet\bullet})^{\bullet} = (U^{\bullet\bullet} \multimap V)^{\bullet}$$

Lemma 4.14. *If $\perp$ is precise and stable, then $\mathbb{T}(\mathbf{C})$ is a SMCC, with the structure defined in lemma 4.12.*

4.4 Models from the book

We present here three famous models of linear logic, which all revealed to be an instance of double-glueing. Due to the lack of space, we will not recall their definitions, though we give here a brief glimpse.

- Coherent spaces **Coh** [7] are about *cliques* and stable relations;
- Finiteness spaces **Fin** [4] rely on the notion of *finitary sets* and finitary-preserving relations;
- Phase semantics [7] considers *closed subsets* of a commutative monoid.

The latter two already include a notion of orthogonality in their base definition, so the relationship with double-glueing is not surprising. The link between coherent spaces and orthogonalities is considered to be folklore amongst specialists, though it has seldom been formally written in articles.

Definition 4.15. Let R be a set, and let $u, x \in \mathcal{P}(R)$. We define two orthogonalities:

$$u \perp_R^1 x := |u \cap x| \leq 1$$
$$u \perp_R^\omega x := |u \cap x| < \infty$$

Lemma 4.16. *We consider* **Rel** *the category of sets and relations, which is compact closed. Then* $\mathbf{Coh} = \mathbb{T}(\mathbf{Rel})$ *for* $\perp^1$ *and* $\mathbf{Fin} = \mathbb{T}(\mathbf{Rel})$ *for* $\perp^\omega$.

Remark. Both orthogonalities are not stable, hence theorem 4.14 is not directly applicable. Yet one can explain the structure lift by in-depth arguments. Another way, as in [17], is to generalize relations $f \subseteq R \times S$ in **Rel** to matrix-like functions $f : R \times S \to \overline{\mathbb{N}}$.

The positive slack corresponding to **Coh** is known as configuration spaces in [14].

Definition 4.17. Let $\mathcal{M}$ be a commutative monoid, and let $\perp\!\!\!\perp \subseteq \mathcal{M}$. For any $u, x \in \mathcal{M}$, we say that $u \perp x$ whenever $ux \in \perp\!\!\!\perp$.

Lemma 4.18. *Let us consider* **M** *the one-object compact closed category that arises from $\mathcal{M}$. Then the multiplicative part of phase semantics on $(\mathcal{M}, \perp\!\!\!\perp)$ is $\mathbb{T}(\mathbf{M})$.*

4.5 Focussed orthogonalities

Once again, as for the enriched case, there are orthogonalities which are simple and particularly well-behaved w.r.t. double-glueing. They are defined the same way.

Definition 4.19. We say that $\perp$ is focussed if there exists a pole $\perp\!\!\!\perp \subseteq \mathbf{C}(1, \perp)$ such that for any $u : \mathbf{C}(1, R)$ and $x : \mathbf{C}(R, \perp)$, $u \perp_R x$ iff $u; x \in \perp\!\!\!\perp$.

Example. The orthogonality from phase semantics is focussed, as is the one from ludics [6].

Lemma 4.20. *Suppose $\perp$ is focussed. Then $\perp$ is precise, stable and moreover, any morphism is focussed.*

Remark. In particular, when $\perp$ is focussed, $\mathbb{S}^{\downarrow}(\mathbf{C}) = \mathbb{S}^{\uparrow}(\mathbf{C}) = \mathbb{S}(\mathbf{C})$.

4.6 The intuitionistic case

As a special case of symmetric monoidal closed categories, and as a famous model of lambda-calculus, one must consider the cartesian closed categories. Nevertheless, there is a important clash that appears when applying double-glueing to CCCs, not unlike the degeneracy that occurs when violently requiring the classical isomorphism $A \cong (A \Rightarrow \perp) \Rightarrow \perp$ for any A.

Lemma 4.21. *Let $\top$ be a terminal object in $\mathbb{S}(\mathbf{C})$. Then its set of counter-proofs is empty.*

Lemma 4.22. *Suppose $\mathbf{C}$ is a cartesian closed category, hence in particular a symmetric monoidal closed category. Then $\mathbb{S}(\mathbf{C})$ preserves the CCC structure iff $\perp$ is the empty orthogonality.*

5 Exponential modality

Up to now, we only studied the exponential-free fragment of linear logic. As usual, the real problem arises as soon as one tries to leave this enchanted kingdom by adding exponential modalities.

5.1 Linear distributions

Contrary to the MALL fragment, there is no unique way to lift exponentials from the base category to the glued category. Even worse, there are several axiomatizations of the exponential out there which are not exactly equivalent, at least from the point of view of their internal language [16].

We will stick to the choice of the original paper, that is, linear categories. For the sake of clarity, we roughly recall here the structure we need for a linear category.

Definition 5.1 (In a nutshell)**.** A linear category is an SMCC $(\mathbf{C}, \otimes, 1)$ with a symmetric monoidal comonad, i.e.

- A functor $! : \mathbf{C} \to \mathbf{C}$ and two natural transformations, dereliction $\varepsilon_R : \mathbf{C}(!R, R)$ and digging $\delta_R : \mathbf{C}(!R, !!R)$;

equipped with well-behaved comonoids, i.e.

- for any object R, a weakening $e_R : \mathbf{C}(!R, 1)$ and a contraction $d_R : \mathbf{C}(!R, !R \otimes !R)$;

and a monoidal distribution:

- $m_1 : \mathbf{C}(1, !1)$ and $m_{R,S} : \mathbf{C}(!R \otimes !S, !(R \otimes S))$ natural in R and S.

These morphisms are subject to a herd of coherence diagrams. The bravehearted reader may refer to [2].

The various ways to lift exponentials from a linear category to its double-glueing are represented through the choice of a linear distribution. Unlike the original paper, we generalized this structure to include *non-uniform* exponentials, inspired by game semantics. Non-uniform exponentials are able to use a whole array of linear strategies to answer the opponent, while uniform ones only keep repeating the same strategy.

Definition 5.2. A linear distribution on a linear category $\mathbf{C}$ is a family of functions

$$\kappa_R : \mathcal{P}(\mathbf{C}(1, R)) \to \mathcal{P}(\mathbf{C}(1, !R))$$

subject to the following requirements, for any $U, U' \subseteq \mathbf{C}(1, R)$, $V \subseteq \mathbf{C}(1, S)$ and $f : \mathbf{C}(R, S)$:

Monotonicity	$U \subseteq U'$	$\Rightarrow$	$\kappa_R(U) \subseteq \kappa_R(U')$
Naturality	$\kappa_R(U); !f$	$\subseteq$	$\kappa_S(U; f)$
Comonad compatibility	$\kappa_R(U); \varepsilon_R$	$\subseteq$	U
	$\kappa_R(U); \delta_R$	$\subseteq$	$\kappa_{!R}(\kappa_R(U))$
Comonoid compatibility	$\kappa_R(U); e_R$	$\subseteq$	$\{\mathrm{id}_1\}$
	$\kappa_R(U); d_R$	$\subseteq$	$\kappa_R(U) \otimes \kappa_R(U)$
Monoidality	m_1	$\in$	$\kappa_1(\{\mathrm{id}_1\})$
	$\kappa_R(U) \otimes \kappa_S(V); m_{R,S}$	$\subseteq$	$\kappa_{R \otimes S}(U \otimes V)$

We say a linear distribution is *uniform* (or *pointwise*) when $\kappa_R(U) = \{\hat{\kappa}_R(u) \mid u \in U\}$ for some natural transformation $\hat{\kappa}_R : \mathbf{C}(1, R) \to \mathbf{C}(1, !R)$.

Lemma 5.3. *For any linear category, there exists a canonical uniform linear distribution:*

$$\hat{\kappa}_R(u) : \; 1 \xrightarrow{m_1} !1 \xrightarrow{!u} !R$$

Remark. In almost any model from the litterature, the linear distribution used is indeed the canonical one.

From now on, we suppose that $\mathbf{C}$ is linear and has a linear distribution κ.

5.2 Slack categories

We are now able to lift exponentials onto the slack category.

Lemma 5.4. *Suppose ε_R, e_R and d_R are positive w.r.t. $\kappa_R(U)$ for any $U \subseteq \mathbf{C}(1, R)$. Then $\mathbb{S}(\mathbf{C})$ is a linear category, with the following inductive structure, for $A = (R, U, X)$:*

$$\frac{\tilde{u} \in \kappa(U)}{\tilde{u} \Vdash^p !A} \qquad\qquad \frac{x \Vdash^o A}{\varepsilon_R; x \Vdash^o !A} \quad \frac{\chi \Vdash^o 1}{e_R; \chi \Vdash^o !A} \quad \frac{h \Vdash^o !A \otimes !A}{d; h \Vdash^o !A}$$

Remark. The object $!A$ is not polarized: though we would like it to be positive for any A, it is not in general.

5.3 Tight categories

As stated in the previous remark, in order to fit an exponential object into the tight category, we need to close its components on both sides. We must pay attention, as the order matters.

Definition 5.5. For any object $A = (R, U, U^\bullet)$ in $\mathbb{T}(\mathbf{C})$, we define its exponential as:

$$!_{\mathbb{T}(\mathbf{C})}A = \uparrow\downarrow(!_{\mathbb{S}(\mathbf{C})}A) = (!R, \kappa_R(U)^{\bullet\bullet}, \kappa_R(U)^\bullet)$$

Lemma 5.6. *Suppose $\perp$ is precise, stable and that the following morphisms are positive w.r.t. any $\kappa(U)$: ε_R, δ_R, e_R, d_R and any $!f$. We also suppose that $m_{R,S}$ is positive. Then $\mathbb{T}(\mathbf{C})$ is a linear category, with the exponential as in definition 5.5.*

Conclusion

As such, double-glueing relies on a whole set on complicated axioms; in practice, when designing new models through this construction, we often restrict to cases where everything is simpler, that is to know, focussed orthogonalities.

This is also a robust process, as it easily scales to weaker models such as enriched models, and in particular, polar and dialogue categories. This may indicate that double-glueing is something generic in the quasi-linear world. Conversely, double-glueing is rather badly behaved with respect to intuitionistic logic, whose behaviour is closer to simple-glueing, i.e. forward preservation only: this is typical of the assymetry pervasive in an intuitionistic realm.

The creation of polarization out of the blue is something quite surprising, to say the least. It may be a point to be developped, in order to get a better grasp of the foundations of the polarized structure. Up to now, polarized logic is still crippled with remnants of its origin that date back to game semantics.

On different grounds, there is still things to say about exponentials. The linear category paradigm is not satisfactory enough, because it is exaggerately *ad-hoc*. The various attempts at lifting exponentials to the adjunction model setting all failed. Nonetheless, a particular definition seems promising. Anyway, this could lead to two different discoveries, which are not unrelated: on the one hand, resolving the polarization problem of exponentials; on the other hand, refining the either the adjunction model or the logic itself.

Finally, a whole new world of models are waiting for us to be discovered, either through direct application of double-glueing, or by stirring it with other bits of (classical) realizability.

References

[1] Nick Benton. "A Mixed Linear and Non-Linear Logic: Proofs, Terms and Models (Extended Abstract)". In: *Computer Science Logic*. Ed. by Leszek Pacholski and Jerzy Tiuryn. Vol. 933. Lecture Notes in Computer Science. Springer, 1994, pp. 121–135.

[2] Gavin Bierman. "What is a categorical model of Linear Logic?" In: *Typed Lambda Calculi and Applications '95*. Ed. by Mariangiola Dezani and Gordon Plotkin. Vol. 902. Lecture Notes in Computer Science. Springer, Apr. 1995.

[3] Jeff Egger, Rasmus Ejlers Møgelberg, and Alex Simpson. "Enriching an effect calculus with linear types". In: *Proceedings of the 23rd CSL international conference and 18th EACSL Annual conference on Computer science logic*. CSL'09/EACSL'09. Berlin, Heidelberg: Springer-Verlag, 2009, pp. 240–254. ISBN: 3-642-04026-8, 978-3-642-04026-9.

[4] Thomas Ehrhard. "Finiteness spaces". In: *Mathematical Structures in Computer Science* 15.4 (2005), pp. 615–646.

[5] Jean-Yves Girard. "A New Constructive Logic: Classical Logic". In: *Mathematical Structures in Computer Science* 1.3 (1991), pp. 255–296.

[6] Jean-Yves Girard. "From foundations to ludics". In: *Bulletin of Symbolic Logic* 9.2 (2003), pp. 131–168.

[7] Jean-Yves Girard. "Linear Logic: its syntax and semantics". In: *Advances in Linear Logic*. Cambridge University Press, 1995, pp. 1–42.

[8] Jean-Yves Girard. "Locus Solum: From the rules of logic to the logic of rules". In: *Mathematical Structures in Computer Science* 11.3 (June 2001), pp. 301–506.

[9] Martin Hyland and Andrea Schalk. "Glueing and orthogonality for models of linear logic". In: *Theor. Comput. Sci.* 294.1/2 (2003), pp. 183–231.

[10] Olivier Laurent. "Polarized games". In: *Annals of Pure and Applied Logic* 130.1–3 (Dec. 2004), pp. 79–123.

[11] Paul Blain Levy. "Call-by-push-value". PhD thesis. 2001.

[12] Paul-André Melliès. "Categorical semantics of linear logic". In: vol. 27. Panoramas et Synthèses. Société Mathématique de France, 2009. Chap. 1, pp. 15–215.

[13] Paul-André Melliès and Nicolas Tabareau. "Resource modalities in tensor logic". In: *Annals of Pure Applied Logic* 161.5 (Feb. 2010), pp. 632–653.

[14] Paul-André Melliès, Nicolas Tabareau, and Christine Tasson. "An explicit formula for the free exponential modality of linear logic". In: *Automata, Languages and Programming*. Vol. 5556. Lecture Notes in Computer Science. Springer, July 2009, pp. 247–260.

[15] Paul-André Melliès and Peter Selinger. *Games are continuation models!* 2001.

[16] Valeria De Paiva. *Categorical Semantics of Linear Logic For All*. 2006.

[17] Nicolas Tabareau. "Modalités de ressources et contrôle en logique tensorielle". PhD thesis. 2008.